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FORWARD MATHEMATICS

When the philosophy of secondary education changed from an education for the aristocracy to an education for the masses, mathematics began to lose its importance as a secondary school subject. From the outset, the mathematician attempted to counteract this trend without success.

In later years, the mathematician has attempted to revamp the subject matter of the secondary mathematical program to harmonize with the modern theories and practices of secondary education. This new approach is apparently bringing a slight ray of hope in some quarters. However, satisfactory objectives are far from being achieved, for many students are still coming to college without the necessary preparation to pursue successfully a course in college algebra.

The slogan, *Mathematics for Defense*, together with existing conditions, has given more impetus to the study of mathematics than anything that has taken place since mathematics lost its important position. The effect of this stimulus may be realized to some extent in noting what has taken place at Louisiana State University. Even though the general enrollment dropped 10 per cent this year, the freshman mathematics enrollment increased 30 per cent, which represents the addition of 600 students. It must be realized that this impetus, as a lasting good for the cause of mathematics, is a false one.

In the first place, this incentive alone will not gain for mathematics its former position, because it will be removed at the completion of the war. However, comparable motives may be instituted with respect to peace-time activities that would keep this type of incentive alive.

Secondly, those who study mathematics due to these conditions do so entirely from a practical point of view. It is well and good for individuals to study mathematics for its utilitarian value because it is a highly practical subject. Nevertheless, this attitude alone will not produce students of advanced mathematics from which the greatly needed teachers are to be drawn.

Just as soon as the human race enters upon one period, it must begin to plan for the next; otherwise, the latter state will be more chaotic than the former. Our country is at war. An all-out offensive and defensive battle must be waged by the entire nation if the government is to bring the conflict to a successful close. However, not all persons are useful in the armed forces. Neither will the remainder find their services needed in the production and planning phase. It is necessary to the welfare of the country for those who are not needed elsewhere to work toward the improvement of existing conditions. The cause of secondary mathematics still needs attention, both from the standpoint of teacher-training and curriculum-making. Let us move forward from our present vantage point rather than lose the little ground that has been gained.

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The Theorem of Morley

By J. W. Peters
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If the internal trisectors of the interior angles of a triangle A,B,C, be drawn and if those trisectors adjacent to AB meet at R, and those adjacent to BC meet at P, and those adjacent to AC meet at Q, then P,Q,R form an equilateral triangle.

This is the theorem of Morley—a simple but nonetheless elegant statement about the triangle. Discovered by an American about forty years ago, the theorem has been rediscovered several times since and has appeared in mathematical journals on both sides of the world. But it is in America that I believe it is least known.

Frank Morley, for whom the theorem is named, was professor of mathematics at the Johns Hopkins University from 1900 until his retirement in 1928. He discovered the theorem about 1900, the exact date is uncertain for he did not publish his result. Fortunately, he did mention the theorem in his lectures to his students. The theorem arose as a by-product of some of his work on the cardioid. A cardioid is generated by a point M of a circle of radius r that rolls without slipping on a fixed circle of equal radius. The center of the fixed circle is called the center of the cardioid. If we are given three lines forming a triangle A,B,C, then the center of a variable cardioid inscribed in the triangle will lie on the three line segments PQ, QR, RP. When the center is at P, Q, or R, then BC, CA, or AB respectively is the double tangent of the cardioid. Professor Morley showed that the angles of the triangle P,Q,R, are all equal and that RA,QA are the trisectors of the interior angle at A, that RB, PB are the trisectors of the interior angle at B, and that PC,QC are the trisectors of the interior angle at C. As I have already mentioned these results were not published at that time.

So far as I can determine, the first time the theorem was mentioned in the literature of mathematics was when it was proposed as a problem by E. J. Ebden, in *Mathesis*[1] in 1908 and somewhat later in the *Educational Times.*[2] A solution involving trigonometry by Delahaye and H. Lez[3] appeared on page 138 of *Mathesis* for the year 1908. Three solutions to the problem appeared in the *Educational Times*. First there was a trigonometrical solution by M. Satyanarayana,[4] then two solutions depending on plane geometry alone, one by Naraniengar[5]

and the other by W. F. Beard.[6] An examination shows that the two geometrical proofs are quite similar. For the next five years the theorem went apparently unnoticed.

On November 14, 1913, F. G. Taylor and W. L. Marr presented a paper to the Edinburgh Mathematical Society on the *Six Trisectors of Each of the Angles of a Triangle*, in which they proved the above theorem and arrived at some further results all of which they thought were new. After their paper was read, several persons who had been either students of Morley's or who had had direct contact with him, pointed out his discovery of the theorem. When Taylor and Marr published their paper,[7] they gave Professor Morley credit for the discovery of the theorem and in addition to their own proof of the theorem included another proof by W. E. Phillips. Phillips' proof depends only on a knowledge of plane geometry and while it is long it is not excessively complicated. Marr followed this paper with several others relating to the theorem but after a short time the topic disappeared again from the magazines.

In 1921, the subject was revived in the French magazines. B. Niewenglowski,[8] in *L'Enseignment Mathematique*, gave the first indirect geometrical proof of the theorem. By an indirect proof is meant a proof in which one starts with the equilateral triangle P,Q,R, and shows how to construct a triangle A,B,C, such that QA, RA are the trisectors of the interior angle at A; PB, RB are the trisectors of the interior angle at B; and PC, QC are the trisectors of the interior angle at C. While this was a new proof of the theorem, Niewenglowski recognized the fact that the theorem had been proved before.

In the same year, R. Marcolongo[9] proposed the theorem as a problem in the Italian journal, *Periodico di Mathematicke*. He must have known that it had been solved for he called the theorem by Morley's name. In the same volume of that journal appeared a trigonometrical solution by G. B. Zecca.[10] An editorial note attached to this solution mentions a geometric solution by Signor Lampariello which was quite complicated and not published. The Italians seemed content to leave the theorem at this stage, for I found no further references to Morley's theorem in the Italian journals.

During 1923, three interesting papers concerning the theorem appeared. First J. M. Child[11] gave a direct geometrical proof which is a slight refinement of the earlier proofs of Naraniengar and Beard. Just how much he knew about these earlier proofs is difficult to say. Then R. Bricart gave two proofs of the theorem. His first proof[12] is based on the concepts of elementary geometry, while the second[13] is an analytical proof using complex numbers. Finally, J. Neuberg[14] in

a paper in *Mathesis* considered the trisectors of the exterior angles of the triangle. He included in his paper some theorems by the Belgian mathematician, R. Goormathigh. Their chief result was that the triangles A,B,C and P,Q,R are inscribed in an anallagmatic cubic curve in normal coordinates.

It was in 1924 that Professor Morley[15] published his only paper on the theorem. The paper is entitled *On the Intersections of the Trisectors of the Angles of a Triangle* and appeared in the *Journal of the Mathematical Association of Japan for Secondary Education.* The only place that one can see this magazine in this country is at Brown University. Naturally one wonders why Professor Morley chose such an inaccessible journal. The explanation is simple, once one sees the paper. Apparently Professor T. Hayashi, of the Tohoku Imperial University, inquired of Professor Morley where he could find Morley's proof of the theorem. It was Morley's reply to this inquiry that was published in the Japanese journal. Morley first stated that he never published a proof of the theorem. Then he proceeded to explain how he was led to the discovery of the theorem by the consideration of certain chains of theorems that he discussed in his memoir in the first volume of the *Transactions of the American Mathematical Society.* Finally, he presented his proof of the thorem. In a later paper[16] on *Reflexive Geometry,* he mentioned the theorem again and then added this modest footnote: "This theorem which I obtained in this way long ago has excited much interest." In his book on *Inversive Geometry*[19] he sketched his proof and dismissed the theorem in about a dozen lines.

For three years, from 1924 to 1937, the theorem disappeared from the journals again. Then in 1927, Phillip Franklin[18] discussed the relation between the Simson Lines, the Morley Triangles, and the Three Cusped Hypocycloids of a Triangle. Since this is one of the two references to Morley's theorem that I have seen in American texts, it seems decidedly unfortunate that Franklin should attribute the theorem to 'John' Morley. In his book on *Modern Geometry,* R. A. Johnson discussed Morley's theorem, gave the proof by W. E. Phillips, and discussed the work of Taylor and Marr.

The paper by Lob and Richmond[17] discussed a general method that is applicable to the proofs of theorems concerning triangles, and the authors illustrated their work by proving the theorem of Morley. The papers by Gambier[21] and D. Chiocas[22] I have not seen. The chief interest that the latter holds for me is that the theorem reached the Balkan peninsula.

In 1933 the theorem came to the attention of the Germans. First a variation of the theorem was proposed as a problem in the *Zeitschrift*

fur Mathematischen und Naturwissenschaftlichen Unterricht[23] in 1933 and a solution appeared in 1934. Attached to the solution were several references to Morley's theorem. In 1937, K. Lorenz[24] discovered the theorem anew and gave a trigonometric solution. The paper by M. Zacharias[25] related Morley's theorem to the configuration of 12 points and 16 lines, with 3 points on each line and 4 lines on each point. The paper by J. van Ijzeren[26] is written in Dutch which I cannot read, but I gather that he attributed the theorem to Karl Runge and did not mention Morley. H. v. Kaven[27] and J. Mahrenholz[28] published short sketches of the history of the theorem. An editorial note at the close of the paper by Mahrenholz brought up again Runge's connection with the theorem and the editor claimed he could not find where Runge ever mentioned the theorem. Finally, in 1939, J. C. Hofman[29] gave a "new" proof of Morley's theorem. It developed that the proof was a variation of that given by Niewenglowski eighteen years previously.

Once more the theorem returned to England, when in 1938 W. J. Dobbs[30] published an article on the six trisectors of the angles of a triangle, and found twenty-seven triangles of which eighteen were equilateral. In the same journal, one hundred thirty two pages further on there is a little note by W. L. Marr calling attention to the fact that he and Taylor had discussed the problem a quarter century before.

And thus there arises the question, "Has this theorem which has been discovered at least three times during the past forty years and which has been discussed in so many different places ever been discovered before the beginning of the twentieth century?" So far the answer to this question seems to be "No", but the theorem is so simple that I can scarcely believe that the answer is true.

In the remainder of this paper I shall sketch several different types of elementary proofs of the theorem of Morley. The first proof is the one given by J. M. Child.

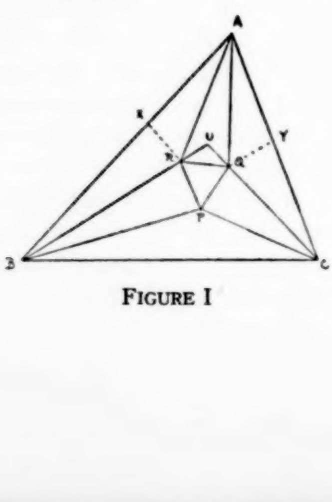

FIGURE I

Let the trisectors adjacent to BC meet in P and let the other two trisectors of B and C meet in U.

Then $\angle BPC = \pi - (B/3) - (C/3)$.

Construct R so that $\angle BPR = (\pi/3) + (C/3)$

and construct Q so that $\angle CPQ = (\pi/3) + (B/3)$.

Then $\angle RPQ = \pi/3$.

In triangle BRP, $\angle BRP = (\pi/3) + (A/3)$

and in triangle CQP, $\angle CQP = (\pi/3) + (A/3)$.

Thus PR and PQ are equally inclined to the sides BU and CU of the triangle BCU for which P is the incenter. Therefore $PR = PQ$ and the triangle PQR is equilateral.

We must yet show that RA and QA are the trisectors of A. To this end choose X on AB so that $BX = BP$ and choose Y on CA so that $CP = CY$. Then by similar triangles $XR = RP$ and $YQ = PQ$, and so $XR = RQ = YQ$. Furthermore

$$\angle XRQ = \pi - (2A/3) \quad \text{and} \quad \angle YQR = \pi - (2A/3).$$

Thus XR and YQ are equally inclined to RQ and the four points X,R,Q,Y are on a circle. Since the chords XR, RQ, YQ are equal, they subtend equal arcs on this circle. Each of these arcs will subtend an angle with vertex on the circumference equal to

$$\tfrac{1}{4}[2\pi - 2\pi + (4A/3)] = A/3.$$

Therefore arc $XRQY$ subtends angle A. Thus angle A is on the circle and AR and AQ trisect A.

The following proof is a variation of that given by Niewenglowski and that given by Hofman. The proof is the so-called indirect type.

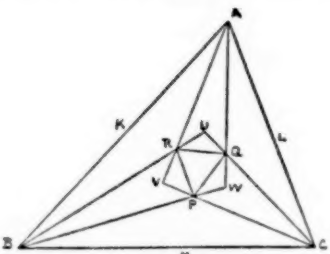

FIGURE II

Consider the equilateral triangle P,Q,R, and three acute angles α,β,γ, all different from zero and subject to the condition that $\alpha+\beta+y=\pi/3$. Construct the points U,V,W so that

$$\angle QRU=\pi/3-\alpha, \quad \angle PRV=\pi/3-\beta, \quad \angle PQW=\pi/3-\gamma,$$

$$\angle RQU=\pi/3-\alpha, \quad \angle RPV=\pi/3-\beta, \quad \angle QPW=\pi/3-\gamma.$$

Let RV and QW intersect in A, RU and PW intersect in B, and VP and UQ intersect in C. It is easy to show that these pairs of lines can be parallel only if α,β, or γ is zero, and this we have excluded.

$$\angle ARQ=\pi-\angle PRQ-\angle PRV=(\pi/3)+\beta,$$

and similalry $\angle AQR=(\pi/3)+\gamma$. Therefore $\angle RAQ=\alpha$.

In like manner one shows that $\angle RBP=\beta$ and that $\angle QCP=\gamma$.

Construct the points K,L,M so that

$$\angle KAR=\angle QAL=\alpha, \quad \angle KBR=\angle PBM=\beta, \quad \text{and} \quad \angle MCP=\angle QCL=\gamma.$$

We must now prove that $\angle AKB=\angle BMC=\angle CLA=\pi$. First considering the angles with vertices at R, we find that $\angle BRA=(2\pi/3)+\gamma$. Considering the quadrangle A,K,B,R, we find that $\angle AKB=\pi$. Similar proofs show that $\angle BMC=\angle CLA=\pi$. Thus we now have the theorem of Morley.

This last proof, involving trigonometry, follows the general pattern of all the trigonometrical proofs given, and as I give it below it probably contains a little bit of all of those proofs that I read.

Refer to figure I again, and let $A=3\alpha$, $B=3\beta$, $C=3\gamma$. Then $\angle BPC=120°+\alpha$, $\angle CQA=120°+\beta$, $\angle ARB=120°+\gamma$. From the triangle CPQ we have that

$$CQ=\frac{b\sin\alpha}{\sin(120°+\beta)} \quad \text{and} \quad CP=\frac{a\sin\beta}{\sin(120°+\alpha)},$$

where a and b are the sides of the triangle A,B,C opposite the angles A,B,C respectively.

Making use of the equation

$$\frac{b}{a}=\frac{\sin 3\beta}{\sin 3\alpha}$$

and of the following identities:

$$\sin 3\alpha=4\sin\alpha\cdot\sin(60°-\alpha)\cdot\sin(60°+\alpha),$$

$$\sin 3\beta=4\sin\beta\cdot\sin(60°-\beta)\cdot\sin(60°+\beta);$$

we have $\dfrac{CQ}{CP}=\dfrac{\sin(60^\circ-\beta \cdot \sin(60^\circ+\beta)\cdot\sin(120^\circ+\alpha)}{\sin(60^\circ-\alpha)\cdot\sin(60^\circ+\alpha)\cdot\sin(120^\circ+\beta)}$.

Since α and β are both acute angles, $\sin(120^\circ+\alpha)=\sin(60^\circ-\alpha)$ and $\sin(120^\circ+\beta)=\sin(60^\circ-\beta)$, and we have

$$\frac{CQ}{CP}=\frac{\sin(60^\circ+\beta)}{\sin(60^\circ+\alpha)} .$$

But

$$\frac{CQ}{CP}=\frac{\sin\angle CPQ}{\sin\angle CQP} .$$

Since $(60^\circ+\beta)+(60^\circ+\alpha)=180^\circ-\gamma$, and since

$$\angle CPQ+\angle CQP=180^\circ-\gamma,$$

we must conclude that $\angle CPQ=60^\circ+\beta$ and $\angle CQP=60^\circ+\alpha$.

Similarly we find that

$$\angle AQR=60^\circ+\gamma, \qquad \angle ARQ=60^\circ+\beta,$$
$$\angle BRP=60^\circ+\alpha, \qquad \angle BPR=60^\circ+\gamma.$$

Checking the sums of the angles with vertices at P,Q,R respectively, we find that the interior angles of the triangle P,Q,R are all equal to 60°.

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Analytic Geometry of the Triangle

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Introduction. A number of papers* have been published in the past few years in which the properties of the triangle are developed by means of the theory of functions of a single complex variable. Normal trilinear coordinates have been used recently only to a limited extent.† It is the purpose of this paper to point out that the latter method furnishes a ready means of proving analytically a great many of the properties of the triangle ordinarily proved synthetically in courses in Euclidean geometry.

Let A_i, α_i, a_i $(i=1,2,3)$ be respectively the vertices, the corresponding angles and the lengths of the opposite sides of the triangle $A_1A_2A_3$. Consider a system of normal trilinear coordinates, in which the coordinates (x_1, x_2, x_3) of any point x in the plane of the triangle are proportional to the distances of x from the sides A_2A_3, A_3A_1, A_1A_2 of the triangle. Since many important points connected with the triangle appear in groups of three, we shall adopt the following notation for the coordinates of such points. For a point whose name contains a variable subscript or superscript, as P_i or $Q^{(j)}$, the first coordinate is to have the same subscript as the subscript or superscript of the name, while the order of the subscript of the other two coordinates is to be such as to preserve the cyclic order (1 2 3). For example, the coordinates of the point P_i shall be written in the order (x_i, x_j, x_k) where the order of $(i\ j\ k)$ shall be in the cyclic order (1 2 3).

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Evidently then $(ax) \equiv a_1x_1+a_2x_2+a_3x_3=2\Delta$, where Δ is the area of the triangle $A_1A_2A_3$. Salmon* shows that any straight line may be represented by a linear homogeneous equation of the form

$$(mx) \equiv m_1x_1+m_2x_2+m_3x_3=0.$$

In particular the equation of the side A_jA_k of the triangle is

$$x_i=0(i,j,k)=1,2,3;\ i\neq j\neq k\neq i)$$

and the equation of the line at infinity is $(ax)=0$. Three lines $(mx)=0$, $(nx)=0$, $(rx)=0$ are concurrent if the determinant of their coefficients

$$(1) \qquad |m\ n\ r| \equiv \begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = 0.$$

Similarly three points y, z, w are collinear if the determinant of their coordinates $|y\ z\ w|=0$. Two lines $(mx)=0$ and $(nx)=0$ are parallel if they are concurrent with the line at infinity, i. e., if $|m\ n\ a|=0$. Salmon† also shows that two lines $(mx)=0$ and $(nx)=0$ are perpendicular if

$$(2) \qquad (mn)-\Sigma(m_jn_k+m_kn_j)\cos\alpha_i \equiv m_1n_1+m_2n_2+m_3n_3$$
$$-(m_2n_3+m_3n_2)\cos\alpha_1-(m_3n_1+m_1n_3)\cos\alpha_2-(m_1n_2+m_2n_1)\cos\alpha_3=0.$$

The equation of the line through two points y and z is $|x\ y\ z|=0$. The equation of any line through the point of intersection of two lines $(mx)=0$ and $(nx)=0$ is $\rho(mx)+\rho'(nx)=0$, ρ and ρ' constants.

Bisectors. The equation of the internal bisector t_i of the angle α_i is $x_j-x_k=0$. The bisectors of the three angles α_1, α_2, α_3 are concurrent in the incenter $I(1,1,1)$. The equation of the external bisector t_i' of the angle α_i is $x_j+x_k=0$. The internal bisector of α_i and the external bisectors of α_j and α_k, i. e., t_i, t_j', t_k', are concurrent in the excenter $I^{(i)}(-1,1,1)$.

Medians. The mid-point of the side A_jA_k is $A_i'(0,\ a_i\sin\alpha_k/2,\ a_i\sin\alpha_j/2)$, or since $a_1/\sin\alpha_1=a_2/\sin\alpha_2=a_3/\sin\alpha_3$ and since the coordinates are homogeneous, the mid-point is $A_i'(0,a_k,a_j)$. The equation of the median m_i issued from A_i is then $a_jx_j-a_kx_k=0$. The three medians m_1, m_2, m_3 are concurrent in the centroid or median point $M(a_2a_3,\ a_3a_1,\ a_1a_2)$. The exmedian m_i' is defined as the line through A_i parallel to the opposite side A_jA_k. The equation of m_i' is

*G. Salmon, *A treatise on conic sections*, p. 57.
†*Ibid.*, p. 59.

$a_j x_j + a_k x_k = 0$. The median m_i and the exmedians m_j', m_k' are concurrent in the exmedian point $M^{(i)}(-a_j a_k, a_k a_i, a_i a_j)$.

Altitudes. If $(mx)=0$ is perpendicular to $A_j A_k : x_i = 0$, then $m_i - m_j \cos\alpha_k - m_k \cos\alpha_j = 0$. Hence the equation of the altitude h_i issued from A_i is $x_j \cos\alpha_j - x_k \cos\alpha_k = 0$. The three altitudes h_1, h_2, h_3 are concurrent in the orthocenter $H(\sec\alpha_1, \sec\alpha_2, \sec\alpha_3)$.

Circumcircle. The equation of the perpendicular bisector b_i of the side $A_j A_k$ is $(a_j \cos\alpha_j - a_k \cos\alpha_k)x_i - a_j \cos\alpha_i\, x_j + a_k \cos\alpha_i x_k = 0$. The three perpendicular bisectors, b_1, b_2, b_3, of the sides of the triangle are concurrent in the circumcenter $O(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3)$. Salmon* shows that the equation of the circumcircle of the triangle is

$$(3) \qquad \Sigma a_i x_j x_k \equiv a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 = 0,$$

and that the equation of any circle in the plane may be put in the form $\Sigma a_i x_j x_k - (\rho x)(ax) = 0$, where $\rho_i (i=1,2,3)$ is a parameter. The line $(\rho x)=0$ is the radical axis of the generic circle and the circumcircle.

Nine-point Circle. The equation of the nine-point circle of the triangle may be obtained by determining the parameters ρ_1, ρ_2, ρ_3 so that the circle will contain the mid-points A_1', A_2', A_3' of the sides of the triangle. The generic circle will contain A_i' if $a_i - 2(\rho_j a_k + \rho_k a_j) \equiv 0$, whence $\rho_1 = (\cos\alpha_i)/2$. Hence the equation of the nine-point circle is

$$2\Sigma a_i x_j x_k - (x_1 \cos\alpha_1 + x_2 \cos\alpha_2 + x_3 \cos\alpha_3)(ax) = 0.$$

Its radical axis with the circumcircle, $\Sigma x_i \cos\alpha_i = 0$, is called the orthic axis. It is readily seen that the equation of the nine-point circle is satisfied by the coordinates of the foot of the altitude h_i, $H_i(0, \cos\alpha_k, \cos\alpha_j)$, and by the coordinates of the point

$$H_i'(-\cos(\alpha_j - \alpha_k), \cos\alpha_i \cos\alpha_k, \cos\alpha_i \cos\alpha_j).$$

The latter point also lies on the altitude h_i. It will be shown later to be the mid-point of the segment joining the orthocenter H to the vertex A_i. The points $A_i', H_i, H_i' (i=1,2,3)$ are the nine points for which the circle is named.

Pedal Triangle and Circle. The perpendicular dropped from any point $P(y_1, y_2, y_3)$ on the side $A_j A_k$ meets that side in the vertex $P_i(0, y_j + y_i \cos\alpha_k, y_k + y_i \cos\alpha_j)$ of the pedal triangle of the point P

*Ibid., p. 118.

with respect to the triangle of reference. The side P_jP_k of the pedal triangle has the equation

$$-(y_j+y_k\cos\alpha_i)(y_k+y_j\cos\alpha_i)x_i+(y_i+y_k\cos\alpha_j)(y_k+y_j\cos\alpha_i)x_j$$
$$+(y_i+y_j\cos\alpha_k)(y_j+y_k\cos\alpha_i)x_k=0.$$

The circle through P_1, P_2 and P_3 is called the pedal circle of P. Its equation is

$$(ay)\Sigma a_iy_jy_k\Sigma a_ix_jx_k-(ax)\Sigma a_ja_ky_i(y_j+y_k\cos\alpha_i)(y_k+y_j\cos\alpha_i)x_i=0.$$

The pedal triangle of a point P will reduce to a line if P_1, P_2, P_3 are collinear. This condition reduces to $\Sigma a_iy_jy_k=0$, i. e., P is on the circumcircle. In this case the line is called the Simson's line or pedal line of the point P with respect to the triangle and P is called its pole.

If P is at O, the circumcenter, P_i is the mid-point A_i' of the side A_jA_k and the equation of the side P_jP_k of the pedal triangle becomes $-a_ix_i+a_jx_j+a_kx_k=0$. This line is parallel to A_jA_k. The pedal triangle of O is called the medial triangle of $A_1A_2A_3$. The pedal circle of O is the nine-point circle.

If P is at H, the orthocenter, the pedal triangle is known as the orthic triangle. Its vertex P_i is the point H_i and the equation of its side H_jH_k is $-x_i\cos\alpha_i+x_j\cos\alpha_j+x_k\cos\alpha_k=0$. The pedal circle of H is again the nine-point circle.

Incircle. If P is at $I(1,1,1)$, then P_i becomes $I_i(0,\ 1+\cos\alpha_k,\ 1+\cos\alpha_j)$. Letting $2s=a_1+a_2+a_3$, the equation of the side I_jI_k of the pedal triangle of I becomes

$$-a_i(s-a_i)x_i+a_j(s-a_j)x_j+a_k(s-a_k)x_k=0.$$

The pedal circle of I is the incircle of $A_1A_2A_3$. Its equation is

$$4a_1a_2a_3\Sigma a_ix_jx_k-(ax)\Sigma a_i(-a_i+a_j+a_k)^2x_i=0.$$

Excircles. If P is at $I^{(i)}(-1,1,1)$, P_i, P_j, P_k become respectively $I_i^{(i)}(0,\ 1-\cos\alpha_k,\ 1-\cos\alpha_j)$, $I_j^{(i)}(-1+\cos\alpha_k,\ 0,\ 1+\cos\alpha_i)$

$$I_k^{(i)}(-1+\cos\alpha_j,\ 1+\cos\alpha_i,\ 0),$$

where the coordinates take their order from the superscript. The equation of the excircle which is the pedal circle of $I^{(i)}$ is

$$a_1a_2a_3\Sigma a_ix_jx_k-(ax)\left(a_is^2x_i+a_j(s-a_k)^2x_j+a_k(s-a_j)^2x_k\right)=0.$$

Gergonne Points. The equation of the line joining I_i to the vertex A_i is $a_j(s-a_j)x_j-a_k(s-a_k)x_k=0$. The three lines A_1I_1, A_2I_2, A_3I_3 are concurrent in the Gergonne point

$$G\left(a_2a_3(s-a_2)(s-a_3),\ a_3a_1(s-a_3)(s-a_1),\ a_1a_2(s-a_1)(s-a_2)\right).$$

The lines joining the vertices A_i, A_j, A_k respectively to the points $I_i^{(i)}$, $I_j^{(i)}$, $I_k^{(i)}$ have the equations: $a_j(s-a_k)x_j-a_k(s-a_j)x_k=0$, $a_k(s-a_j)x_k+a_isx_i=0$, $a_isx_i+a_j(s-a_k)x_j=0$. These three lines are concurrent in the ex-Gergonne point

$$G^{(i)}\left(-a_ja_k(s-a_j)(s-a_k),\ a_ka_i^2(s-a_k),\ a_ia_j^2(s-a_j)\right).$$

Nagel Point. The equation of the line joining the vertex A_i to the internal point of contact $I_i^{(i)}$ of the excircle relative to A_i is $(1-\cos\alpha_j)x_j-(1-\cos\alpha_k)x_k=0$. The three lines $A_iI_i^{(i)}$, $A_jI_j^{(j)}$, $A_kI_k^{(k)}$ are concurrent in the Nagel point

$$N\left(a_2a_3(s-a_1),\ a_3a_1(s-a_2),\ a_1a_2(s-a_3)\right).$$

The line $I^{(i)}\ I_i^{(i)}$ through the excenter $I^{(i)}$ and perpendicular to the side A_jA_k has the equation $(\cos\alpha_j-\cos\alpha_k)x_i-(1-\cos\alpha_j)x_j+(1-\cos\alpha_k)x_k=0$. The three lines $I^{(1)}I_1^{(1)}$, $I^{(2)}I_2^{(2)}$, $I_3^{(3)}I^{(3)}$ are concurrent in the point $V(1+\cos\alpha_1-\cos\alpha_2-\cos\alpha_3,\ 1-\cos\alpha_1+\cos\alpha_2-\cos\alpha_3,\ 1-\cos\alpha_1-\cos\alpha_2+\cos\alpha_3)$.

Euler Line. The Orthocenter H, the median point M and the circumcenter O are collinear in the *Euler* line those equation is

$$a_1(a_2\cos\alpha_2-a_3\cos\alpha_3)x_1+a_2(a_3\cos\alpha_3-a_1\cos\alpha_1)x_2$$
$$+a_3(a_1\cos\alpha_1-a_2\cos\alpha_2)x_3=0.$$

The incenter I, the Nagel point N and the median point M are collinear in the line whose equation is

$$a_1(a_2-a_3)x_1+a_2(a_3-a_1)x_2+a_3(a_1-a_2)x_3=0.$$

The incenter I, the circumcenter O and the point V are collinear in the line whose equation is

$$(\cos\alpha_2-\cos\alpha_3)x_1+(\cos\alpha_3-\cos\alpha_1)x_2+(\cos\alpha_1-\cos\alpha_2)x_3=0.$$

The equations of the perpendiculars dropped from the vertex A_i on the internal and external bisectors of the angles a_j and α_k are respectively:

$$a_jx_j+(a_k-a_i)x_k=0,\quad (a_j-a_i)x_j+a_kx_k=0,$$
$$a_jx_j+(a_k+a_i)x_k=0,\quad (a_j+a_i)x_j+a_kx_k=0.$$

The feet of these perpendiculars are collinear in the line $-a_ix_i+a_jx_j+a_kx_k=0$, i. e., the side of the medial triangle which is parallel to the side A_jA_k.

Harmonic Properties. The cross ratio of the lines $(mx)=0$, $(nx)=0$ and the two points y and z may be defined as

$$\frac{(my)}{(mz)} \Big/ \frac{(ny)}{(nz)} \quad \text{or} \quad \frac{(my)(nz)}{(mz)(ny)}.$$

The lines are said to separate the points harmonically if the cross ratio has the value -1. The cross ratio of four concurrent lines is defined as the cross ratio of two of the lines and two points, one on each of the other two lines. The cross ratio of four collinear points is defined as the cross ratio of two of them and any two lines, one through each of the other two points. The cross ratio of the two lines: $A_iO : x_j \cos\alpha_k - x_k \cos\alpha_j = 0$, $A_iH : x_j \cos\alpha_j - x_k \cos\alpha_k = 0$ and the two points $I(1,1,1,1)$, $I^{(i)}(-1,1,1)$ is

$$\frac{(\cos\alpha_k - \cos\alpha_j)(-\cos\alpha_j - \cos\alpha_k)}{(-\cos\alpha_k - \cos\alpha_j)(\cos\alpha_j - \cos\alpha_k} = -1.$$

Hence the two lines A_iO and A_iH separate the two points I, $I^{(i)}$ harmonically and the pencil of lines $A_i(OH\ II^{(i)})$ is harmonic.

The internal and external bisectors of an angle of a triangle are separated harmonically by the sides of the triangle issued from the same vertex. The median and exmedian issued from the same vertex of a triangle are separated harmonically by the sides issued from that vertex.

Symmedians. The symmedian s_i of a triangle may be defined as the symmetric of the median m_i with respect to the internal bisector t_i. The equation of the symmedian s_i is $a_kx_j - a_jx_k = 0$. The three symmedians are concurrent in the symmedian point or Lemoine point $K(a_1, a_2, a_3)$. The exsymmedian s_i' may be defined as the harmonic conjugate of the symmedian s_i with respect to the sides A_iA_j and A_iA_k of the triangle. The equation of s_i' is $a_kx_j + a_jx_k = 0$. The symmedian s_i and the excymmedians s_j', s_k' are concurrent in the exsymmedian point $K^{(i)}(-a_i, a_j, a_k)$. The exsymmedians are parallel to the corresponding sides of the orthic triangle.

The harmonic conjugate of the mid-point of a line segment with respect to its end-points is the point of intersection of the line with the line at infinity. The altitude h_i meets the line at infinity, $(ax)=0$, in the point $H_i''(-a_i, \cos\alpha_k, \cos\alpha_j)$. The range of points $(A_iH\ H_i'H_i''$ is harmonic, and H_i', which has already been shown to lie on the nine point circle, is the mid-point of A_iH.

Pole and Polar. The polar of a point y with respect to a curve $f(x)=0$ may be defined as

$$x(\partial f/\partial x) \equiv x_1(\partial f/\partial x_1)_y + x_2(\partial f/\partial x_2)_y + x_3(\partial f/\partial x_3)_y = 0,$$

where the subscripts on the partial derivatives indicate that these derivatives are to be evaluated at the point y. Evidently the polar of a point with respect to any curve is a line. The point is called the pole of the line. The polar of any point $P(y_1, y_2, y_3)$ with respect to the circumcircle of the triangle $A_1A_2A_3$ is

$$(a_2y_3+a_3y_2)x_1+(a_3y_1+a_1y_3)x_2+(a_1y_2+a_2y_1)x_3=0.$$

The polar of the circumcenter O is the line at infinity. The polar of the Lemoine point K with respect to the circumcircle is called the Lemoine axis. Its equation is $a_2a_3x_1+a_3a_1x_2+a_1a_2x_3=0$.

The polar with respect to a curve of a point on the curve is the tangent line at the point. Hence the tangent to the circumcircle at the vertex A_i is $a_kx_j+a_jx_k=0$, i. e., the exsymmedian s_i' of the triangle. The polar of the center of any circle with respect to that circle is the line at infinity. This property may be used to determine the coordinates of the center of the nine-point circle.

The equation of the polar of any point $P(y_1, y_2, y_3)$ with respect to the nine-point circle is $\Sigma(2a_i \cos \alpha_i y_i - a_k y_j - a_j y_k)x_i=0$. This is the line at infinity if $2a_j \cos \alpha_i\, y_i - a_k y_i - a_j y_k = \rho a_i$. Hence the coordinates of the center of the nine-point circle are of the form

$$x_i=a_ja_k(a_j \cos \alpha_j+a_k \cos \alpha_k).$$

The coordinates of the center C satisfy the equation of the Euler line. The cross ratio of the two lines A_iH and A_iM and the two points C and O is -1. Hence the range of points $(HM\ CO)$ is harmonic. The Euler line meets the line at infinity in the point Q whose coordinates are of the form $x_i=a_ja_k(2a_i \cos \alpha_i - a_j \cos \alpha_j - a_k \cos \alpha_k)$. The range $(OH\ CQ)$ is then harmonic and C is the mid-point of the segment HO.

Polar Circle. The circle with respect to which each side of the triangle $A_1A_2A_3$ is the polar of the opposite vertex is called the polar circle. The equation of the generic circle will satisfy the required conditions if $2a_i\rho_jx_i + (a_k - a_i\rho_j - a_j\rho_i)x_j + (a_j - a_i\rho_k - a_k\rho_i)x_k = \rho x_i$. Hence $\rho_i = \cos \alpha_i$ and the equation of the polar circle reduces to $\Sigma a_i \cos \alpha_i\, x_i^2=0$. Its radical axis with the circumcircle, $\Sigma x_i \cos \alpha_1=0$, is the orthic axis. Hence the polar circle, the circumcircle, and the nine-point circle are coaxal. The center of the polar circle is the orthocenter H. Since the Euler line is the line of centers of the polar circle and the circumcircle and since the orthic axis is their radical axis, it follows immediately that these two lines are mutually perpendicular. This may also be shown by substituting from their equations in 2).

Isogonic Centers. Let R_i be the third vertex of the equilateral triangle constructed on A_jA_k and lying entirely outside $A_1A_2A_3$. Its coordinates are $R_i(-\sqrt{3}, \sqrt{3}\cos\alpha_k+\sin\alpha_k, \sqrt{3}\cos\alpha_j+\sin\alpha_j)$. The three lines A_1R_1, A_2R_2, A_3R_3 are concurrent in the first isogonic center R for which $x_i=(\sqrt{3}\cos\alpha_j+\sin\alpha_j)(\sqrt{3}\cos\alpha_k+\sin\alpha_k)$. If R_i' is the third vertex of the equilateral triangle constructed on A_jA_k and on the opposite side of this line from R_i; then the three lines A_1R_1', A_2R_2', A_3R_3' are concurrent in the second isogonic center R' for which $x_i=(\sqrt{3}\cos\alpha_j-\sin\alpha_j)(\sqrt{3}\cos\alpha_k-\sin\alpha_k)$.

Speiker Center. The internal bisector of the angle α_i' of the triangle $A_1'A_2'A_3'$ is parallel to the internal bisector of the angle α_i of $A_1A_2A_3$. Its equation is $a_i(a_j-a_k)x_i+a_j(a_j+a_k)x_j-a_k(a_j+a_k)x_k=0$. The bisectors of α_1', α_2', α_3' are concurrent in the Speiker Center $S(a_2a_3(a_2+a_3),\ a_3a_1(a_3+a_1),\ a_1a_2(a_1+a_2))$. This point lies on the line $\Sigma a_i(a_j-a_k)x_i=0$, which has already been seen to contain the incenter I, the Nagel Point N, and the median point M. The equations of A_iI and A_iS are respectively: $x_j-x_k=0$ and $a_j(a_i+a_j)x_j-a_k(a_i+a_k)x_k=0$. It follows immediately that these two lines separate harmonically the two points M and N. Hence $(IS\ MN)$ is a harmonic range. The line of this range meets the line at infinity in the point Q' for which $x_i=a_ja_k(a_j+a_k-2a_i)$. The points Q' and S are separated harmonically by the lines A_iI and A_iN: $a_j(s-a_k)x_j-a_k(s-a_j)x_k=0$. Hence S is the mid-point of IN.

The Brocard Configuration. We define the ith adjoint circle of the direct group as the circle which passes through A_i and A_j and is tangent to A_jA_k. For the general equation of a circle to be satisfied by the coordinates of A_i and A_j $\rho_i=\rho_j=0$. The equation of the tangent to $\Sigma a_ix_jx_k-\rho_kx_k(ax)=0$ at A_j is $a_kx_i+(a_i-a_j\rho_k)x_k=0$. This is the line $x_i=0$ if $a_i-a_j\rho_k=0$. Hence the equation of the adjoint circle is $(a_j^2-a_i^2)x_kx_i+a_ja_kx_ix_j-a_ka_ix_k^2=0$. In order to find the coordinates of the point of intersection, aside from A_i, of the ith and rth adjoint circles of the direct group, we eliminate x_i between their equations, thus abtaining the condition

$$a_j^2a_k^2x_j^3-a_ja_k(2a_i^2-a_j^2-a_k^2)x_j^2x_k$$
$$-(a_i^2-a_j^2)(a_k^2-a_i^2)x_jx_k^2-a_i^2a_ja_kx_k^3=0.$$

Eliminating x_i between the equation of the circumcircle and the equation of the line at infinity we find that the equation of the product of the isotropic lines through A_i to the circular points at infinity is $a_ja_kx_j^2+(a_j^2+a_k^2-a_i^2)x_jx_k+a_ja_kx_k^2=0$. The left hand side of this equation must be a factor of the left member of the preceding equation.

Hence the equation of the line joining A_i to the fourth point of intersection of the adjoint circles is $a_ja_kx_j-a_i^2x_k=0$. It meets the ith adjoint circles in $\Omega(a_3^2a_1, a_1^2a_2, a_2^2a_3)$, called the first Brocard point. The coordinates of Ω are readily seen to satisfy the equation of the jth adjoint circle also. Hence the three direct adjount circles meet in a point. The lines joining Ω to the vertices A_1, A_2, A_3 are called the direct Brocard rays. The equation of $A_i\Omega$ is $a_ja_kx_j-a_i^2x_k=0$.

The ith adjoint circle of the inverse group is defined as the circle through A_i, A_j and tangent to A_kA_i. Its equation is $(a_i^2-a_j^2)x_jx_k+a_ka_ix_ix_j-a_ja_kx_k^2=0$. The three adjoint circles of the inverse group meet in the second Brocard point $\Omega'(a_1a_2^2, a_2a_3^2, a_3a_1^2)$. It is readily seen that the ith adjoint circle of the direct group and the ith adjoint circle of the inverse group are coaxal with the circumcircle with the side A_iA_j as radical axis. The Brocard ray $A_i\Omega'$ has the equation $a_ja_kx_k-a_i^2x_j=0$.

The line through the circumcenter O and the Lemoine point K is called the Brocard diameter. Its equation is $\Sigma a_ja_k(a_j^2-a_k^2)x_i=0$. The Brocard circle may be defined as the circle through K, Ω, and Ω'. Its equation is

$$a_1^2+a_2^2+a_3^2)\Sigma a_ix_jx_k-(ax)\Sigma a_ja_kx_i=0.$$

The Brocard circle is readily seen to contain the circumcenter O. Its radical axis with the circumcircle is the Lemoine axis. Its center is found, by the methods used in the case of the nine-point circle, to have coordinates of the type $x_i=a_i(2a_j^2a_k^2+a_k^2a_i^2+a_i^2a_j^2-a_i^4)$. The coordinates of the center satisfy the equation of the Brocard diameter thus showing that the latter is actually a diameter of the Brocard circle.

The Brocard rays $A_k\Omega' : a_ia_jx_j-a_k^2x_i=0$ and $A_j\Omega : a_ia_kx_k-a_j^2x_i=0$ meet in the point $B_i(a_1a_2a_3, a_k^3, a_j^3)$ which lies on the perpendicular bisector OA_i' of the side A_jA_k and on the Brocard circle. The three points B_1, B_2, B_3 are the vertices of the first Brocard triangle. The line A_iB_i has the equation $a_j^3x_j-a_k^3x_k=0$. The three lines A_1B_1, A_2B_2, A_3B_3 are concurrent in the point $\Omega''(a_2^3a_3^3, a_3^3a_1^3, a_1^3a_2^3)$, sometimes called the third Brocard point.

The kth adjoint circle of the direct group and the ith adjoint circle of the inverse group, which are respectively tangent to A_iA_j and A_iA_k at A_i, meet again in the finite point $B_i'(a_j^2+a_k^2-a_i^2, a_ia_j, a_ia_k)$. The points B_1', B_2', B_3' are the vertices of the second Brocard triangle. The point B_i' lies on the Brocard circle and the symmedian A_iK of the triangle $A_1A_2A_3$.

The side B_jB_k of the first Brocard triangle has the equation

$$a_i(a_j^2a_k^2-a_i^4)x_i+a_j(a_i^2a_j^2-a_k^4)x_j+a_k(a_i^2a_k^2-a_j^4)x_k=0.$$

The line through the vertex A_i parallel to B_jB_k has the equation

$$a_j(a_k^2-a_i^2)x_j+a_k(a_j^2-a_i^2)x_k=0.$$

The three lines thus obtained are concurrent in the Steiner point S' for which $x_i=a_ja_k(a_i^2-a_j^2)(a_j^2-a_k^2)$. The Steiner point is on the circumcircle of $A_1A_2A_3$. The line B_iB_i' has the equation

$$a_i(a_j^2-a_k^2)x_i-a_j(a_i^2-a_j^2)x_j+a_k(a_i^2-a_k^2)x_k=0.$$

The three lines B_1B_1', B_2B_2', B_3B_3' are concurrent in the median point M.

The ith adjoint circle of the direct group and the kth adjoint circle of the inverse group are both tangent to the side A_jA_k and contain A_i. They meet again in the point $D_i(a_i,\ 2a_k \cos \alpha_i,\ 2a_j \cos \alpha_i)$. The three points D_1, D_2, D_3 are the vertices of the so-called D-triangle. The point D_i is on the median A_iM.

Lemoine Circles. The equation of the line through the symmedian point $K(a_1, a_2, a_3)$ parallel to the side A_jA_k is

$$(a_j^2+a_k^2)x_i-a_ia_jx_j-a_ia_kx_k=0.$$

It meets the sides of $A_1A_2A_3$ in the points $P_i(0, a_k, -a_j)$, $P_i'(a_ia_k, 0, a_j^2+a_k^2)$, $P_i''(a_ia_j, a^2_j+a_k^2, 0)$. The three points P_1, P_2, P_3 are collinear in the line at infinity and the other six points are concyclic in the first Lemoine circle whose equation is

$$(a_1^2+a_2^2+a_3^2)^2\Sigma a_ix_jx_k-(ax)\Sigma a_ja_k(a_j^2+a_k^2)x_i=0.$$

The center of this circle is the mid-point of the line-segment OK. Hence the first Lemoine circle is concentric with the Brocard circle.

The line through the symmedian point K parallel to the exsymmedian s_i' has the equation

$$2a_1a_2a_3x_i+a_k(a_j^2-a_k^2-a_i^2)x_j+a_j(a_k^2-a_i^2-a_j^2)x_k=0.$$

It meets the sides of the triangle in the three points

$$T_i(0,\ -a_j(a_k^2-a_i^2-a_j^2),\ a_k(a_j^2-a_k^2-a_i^2)),$$

$$T_i'(-a_j(a_k^2-a_i^2-a_j^2),\ 0,\ 2a_1a_2a_3),$$

$$T_i''(-a_k(a_j^2-a_k^2-a_i^2),\ 2a_1a_2a_3,\ 0).$$

The points T_1, T_2, T_3 are collinear in the line $\Sigma a_j^2a_k^2 \cos \alpha_i\, x_i=0$, while the other six points lie on the second Lemoine circle

$$(a_1^2+a_2^2+a_3^2)^2\Sigma a_ix_jx_k-4(ax)\Sigma a_j^2a_k^2 \cos \alpha_ix_i=0.$$

The radical axis of the second Lemoine circle with the circumcircle is the line which has just been seen to contain the points T_1, T_2, T_3. This circle is sometimes called the cosine circle. Its center is the Lemoine point K.

Circles of Apollonius. The three circles, each through one vertex of the triangle $A_1A_2A_3$ and the two points in which the bisectors of the angle at that vertex meet the opposite side, are known as the circles of Apollonius. The equation of the Apollonian circle (C_i), through A_i,

$$(a_j^2-a_k^2)\Sigma a_i x_j x_k-(ax)a_i(a_j x_k-a_k x_j)=0.$$

Its center is the point $C_i(0, -a_j, a_k)$, which lies on the side A_jA_k and is the harmonic conjugate with respect to A_j and A_k of the trace on that side of the symmedian issued from the opposite vertex. The radical axis of (C_i) with the circumcircle is the symmedian s_i. The three centers of the three circles of Apollonius are collinear in the Lemoine axis. By subtracting the equations of the Apollonian circles successively from one another it is found that they are coaxal with the Brocard diameter as radical axis. Their points of intersection are called the isodynamic points of the triangle. In order to find the coordinates of these points, we substitute $(ka_1+\cos\alpha_1, ka_2+\cos\alpha_2, ka_3+\cos\alpha_3)$ in the equation of (C_i) and find for k the values: $k=\pm\sin\alpha_i/a_i\sqrt{3}$. Hence the isodynamic points are $J(\sqrt{3}\cos\alpha_1+\sin\alpha_1, \sqrt{3}\cos\alpha_2+\sin\alpha_2, \sqrt{3}\cos\alpha_3+\sin\alpha_3)$ and $J'(\sqrt{3}\cos\alpha_1-\sin\alpha_1, \sqrt{3}\cos\alpha_2-\sin\alpha_2, \sqrt{3}\cos\alpha_3-\sin\alpha_3)$. The isodynamic points separate harmonically the symmedian point K and thecircumcenter O. The vertex D_i of the D-triangle, which has already been seen to lie on one of the direct group and one of the inverse group of adjoint circles, also lies on the circle of Apollonius (C_i).

Cevians. The lines joining the variable point $P(y_1, y_2, y_3)$ to the vertices of the triangle $A_1A_2A_3$ are called the cevians of P with respect to the triangle. The equation of the cevian P_i, i. e., A_iP, is $y_kx_j-y_jx_k=0$. It meets the side A_jA_k in the point $P_i(0, y_j, y_k)$. The points P_1, P_2, P_3 may be called the vertices of the cevian triangle of P. The side P_jP_k of this triangle has the equation $-y_jy_kx_i+y_ky_ix_j+y_iy_jx_k=0$. It meets the side A_jA_k in the point $P_i'(0, -y_j, y_k)$, the harmonic conjugate of P_i with respect to A_j and A_k. The cevian circle, determined by the points P_1, P_2, P_3 has the equation

$$\begin{gathered}2y_1y_2y_3(a_2y_2+a_3y_3)(a_3y_3+a_1y_1)(a_1y_1+a_2y_2)\Sigma a_ix_jx_k\\ -(ax)\Sigma y_jy_k(-a_iy_jy_k(a_ky_k+a_iy_i)(a_iy_i+a_jy_j)\\ +a_jy_ky_i(a_iy_i+a_jy_j)(a_jy_j+a_ky_k)+a_ky_iy_j(a_jy_j+a_ky_k)(a_ky_k+a_iy_i))x_i=0.\end{gathered}$$

If P is the median point M, P_i is the mid point A_i' of the side A_jA_k, the cevian triangle is the median triangle, and the cevian circle is the nine point circle. If P is the orthocenter H, P_i is the foot of the altitude h_i, the cevian triangle is the orthic triangle and the cevian circle is again the nine point circle. If P is the Gergonne point G, the cevian circle is the incircle and P_i is its point of contact with the side A_jA_k. If P is the ex-Gergonne point $G^{(i)}$, the cevian circle is the excircle and P_i is its point of contact with the side A_jA_k.

Let the harmonic conjugate of the cevain P_i with respect to the sides A_iA_j and A_iA_k be called the excevian p_i' of P. Its equation is $y_kx_j+y_jx_k=0$. The cevian p_i and the excevians p_j', p_k' are concurrent in the excevian point $P^{(i)}(-y_i, y_j, y_k)$. The external bisectors and excenters, the exmedians and exmedian points, the exsymmedians and exsymmedian points are respectively the excevians and excevian points of the incenter I, the median point M and the symmedian point K.

The point P_i' lies on the excevian p_i', i. e., the excevian issued from one vertex of the triangle $A_1A_2A_3$, the side opposite that vertex and the corresponding side of the cevian triangle are concurrent.

Trilinear Polar. The points P_1', P_2', P_3' of the preceding section are collinear in the line $\Sigma y_jy_kx_i=0$, called the trilinear polar of P with respect to $A_1A_2A_3$. The trilinear polar of the orthocenter H is the orthic axis and the trilinear polar of the symmedian point K is the Lemoine axis. The symmedian point is the only point whose trilinear polar coincides with its polar with respect to the circumcircle. The trilinear polar of the median point M is the line at infinity. The trilinear polars of the excenters, the exmedian points and the exsymmedian points are respectively the sides of the cevian triangles of the incenter, the median point and the symmedian point.

Associated Points and Lines. The points $P^{(1)}$, $P^{(2)}$, $P^{(3)}$, which are named excevian points above are sometimes called the associated points of P. Dually, let the line $(mx)=0$ meet the side A_jA_k in the point $Q_i(0, -m_k, m_j)$. The line q_i joining Q_i to the vertex A_i has the equation $m_jx_j+m_kx_k=0$. The harmonic conjugate of q_i with respect to the sides A_iA_j and A_iA_k is the line $q_i' : m_jx_j-m_kx_k=0$. The three lines q_1' q_2', q_3' are concurrent in the point $Q(m_2m_3, m_3m_1, m_1m_2)$, the pole of $(mx)=0$ with respect to $A_1A_2A_3$. The line q_i' meets the side A_jA_k in the point $Q_i'(0, m_k, m_j)$. The three points Q_i, Q_j', Q_k' are collinear in the line $(m^{(i)}x)\equiv -m_ix_i+m_jx_j+m_kx_k=0$. The three lines $(m^{(1)}x)=0$, $(m^{(2)}x)=0$, $(m^{(3)}x)=0$ may be called the associated lines of $(mx)=0$. Obviously, the trilinear polars of the associated points of a

given point are the associated lines of the trilinear polar of that point. In particular, the sides of the medial triangle are the associated lines of the line at infinity and the sides of the orthic triangle are the associated lines of the orthic axis.

Isogonal Conjugates. The symmetric r_i of the cevian p_i of a variable point $P(y_1, y_2, y_3)$ with respect to the internal bisector t_i has the equation $y_j x_j - y_k x_k = 0$. The three lines r_1, r_2, r_3 are concurrent in the point $P'(y_2 y_3, y_3 y_1, y_1 y_2)$, called the isogonal conjugate of P. Since the transformation from P to P' is obviously a quadratic involution,* P is the isogonal conjugate of P'. The orthocenter H and the circumcenter O are isogonal conjugates, and the median point M and exmedian point $M^{(i)}$ are respectively the isogonal conjugates of the symmedian point K and the exsymmedian point $K^{(i)}$. The Brocard points are isogonal conjugates and the isogonic centers R and R' are respectively the isogonal conjugates of the isodynamic points J and J'. The vertices A_1, A_2, A_3 are the fundamental points of the transformation P to P' and its inverse, and the sides of the triangle $A_1A_2A_3$ are its principal curves. The six bisectors of the angles $\alpha_1, \alpha_2, \alpha_3$ are invariant under the transformation and the incenter I and the excenters $I^{(1)}, I^{(2)}, I^{(3)}$ are the only fixed points.

Any two points which are isogonal conjugates have the same pedal circle. A line through one vertex of the triangle $A_1A_2A_3$ is carried by the transformation into a line through the same vertex, while a generic line in the plane is carried into a conic through the three vertices. In particular, the line at infinity is transformed into the circumcircle of $A_1A_2A_3$, and the Brocard diameter is transformed into a conic through the points A_1, A_2, A_3, M, H, R and R'.

The generic line $(mx) = 0$ meets the side A_jA_k in the point Q_i whose join to the opposite vertex A_i is the line q_i. The isogonal conjugate of q_i is the line q_i'' : $m_k x_j + m_j x_k = 0$. The line q_i'' meets the side A_jA_k in the point $Q_i''(0, -m_j, m_k)$. The points Q_1'', Q_2'', Q_3'' are collinear in the line whose equation is $m_2 m_3 x_1 + m_3 m_1 x_2 + m_1 m_2 x_3 = 0$, We thus obtain a transformation which carries the line $(mx) = 0$ into its isogonally associated line $\Sigma m_j m_k x_i = 0$. The trilinear polars of two isogonally conjugate points are isogonally associated.

Isotomic Conjugates. The cevian p_i of the variable point P meets the side A_jA_k of $A_1A_2A_3$ in the point P_i. The symmetric of P_i with respect to the mid-point A_i' of A_jA_k is the point $P_i''(0, a_k^2 y_k, a_j^2 y_j)$. The three lines $A_1P_1'', A_2P_2'', A_3P_3''$ are concurrent in the point

*F. H. Daus, Isogonal and isotomic conjugates and their projective generalization, A. M. M., Vol. XLIII (1936), pp. 160-64.

$P''(a_2^2a_3^2y_2y_3,\ a_3^2a_1^2y_3y_1,\ a_1^2a_2^2y_1y_2)$ which is called the isotomic conjugate of P. The Gergonne point G is the isotomic conjugate of the Nagel point N and the third Brocard point Ω'' is the isotomic conjugate of the symmedian point K.

The transformation P to P'' is involutorial quadratic with the vertices A_1, A_2, A_3 as fundamental points and the sides A_2A_3, A_3A_1, A_1A_2 as principal lines. Hence P is the isotomic conjugate of P''. The medians and exmedians of $A_1A_2A_3$ are invariant under the transformation and each of the four points M, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ is its own isotomic conjugate. Again, a line through a vertex is transformed into a line through the same vertex and a generic line is transformed into a conic through the three vertices. In particular, the line $a_1^3x_1+a_2^3x_2+a_3^3x_3=0$ is carried into the circumcircle.

The generic line $(mx)=0$ meets the sides of $A_1A_2A_3$ in three points whose symmetrics with respect to the mid-points of the sides on which they lie are collinear in the line $\Sigma a_i^2m_jm_kx_i=0$ which has been called the reciprocal transversal of $(mx)=0$. The line at infinity is its own reciprocal transversal.

The Form of a Ship

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1. *Mathematical investigations.* Engineers have from time to time presented addresses at mathematical congresses which indicate to some extent the mathematics of ships. Thus Sir William White at Cambridge in 1912 spoke on "The place of mathematics in engineering practice" and at Toronto in 1924 Mr. W. J. Berry spoke on "The influence of mathematics on the development of naval architecture." Much work has been done by the mathematicians since these addresses were delivered. The developments may be classified roughly into (1) applications of dimensional reasoning, (2) solution of simplified problems in which the ship is replaced by a body of simpler form and the ocean by a perfect fluid or fluid of small viscosity (3) the use of numerical or mechanical methods of integration or other types of calculation to obtain approximate results applicable to a ship's actual form.

2. *Dimensional reasoning.* This may have originated when Galileo Galilei compared the areas of the cross sections of the legs of animals with the weights supported.[1] It was used by Sir Isaac Newton[2] in his Principia and by other great mathematicians of the eighteenth century to a limited extent when use was made of certain transformations of variables to simplify equations. New fields of application were opened up when the general equations of viscous flow and of the theory of elasticity were developed by Navier, Poisson, Cauchy, Stokes and others. In 1850 indeed Stokes[3] gave a discussion of the breaking of railway bridges in which he reduced his differential equation to one involving only dimensionless quntities. The value of the method was fully recognized when William Froude,[4] Lord Rayleigh[5] and Osborne Reynolds[6] did their pioneering work. The method is used now in all branches of engineering and has many advocates in this

country of which Dr. Edgar Buckingham[7] of the Bureau of Standards has been the most insistent. He has among other things applied the method to the study of the effect on a ship of the explosion of a shell or torpedo.[8]

3. *Physical and geometrical numbers of nautical interest.* In the application of the foregoing method to the study of the resistamce of ships a dimensionless coefficient which occurs in a mathematical expression for a force or couple acting on the ship is supposed to be a function of certain physical numbers n and of certain geometrical numbers N all of which are independent of one another. The use of a finite number of quantities N really restricts the form of the ship to certain fairly general types but the said number may be made sufficiently large for approximate representation of any desired form.

The physical numbers generally used are:

1°. The Froude number n_F (or F) expressed by the ratio $V/(gL)^{1/2}$ where V is the velocity of the ship in feet per second, L is the length in feet from stern to stem at the water line and g is the acceleration of gravity in feet per second per second. V,L and g may, of course, also be given in the metric system of units as is the custom on the continent of Europe. Naval architects also often prefer to use a speed-length ratio $c=v/L^{1/2}$ where v is the speed in knots.

2°. The Reynolds number n_R expressed by the ratio VL/ν where ν is the kinematic viscosity in square feet per second.

3°. The Mach number n_M expressed by the ratio a/V where a is the velocity of sound in water (or air) in feet per second.

4°. The Thoma number n_T which is of importance in the study of cavitation on the back of a blade of ship's propeller and in the theory of water turbines. Various definitions have been adopted. One writer[9] uses the ratio $(H_a - H_s)/H$ where H is the total head in a turbine, H_a is the water head corresponding to atmospheric pressure minus water tension and H_s is the suction head.

In the theory of the resistance of a ship traveling at a moderate speed the influence of n_M and n_T is small and for similar bodies, i. e., for constant values of the geometrical numbers N a resistance coefficient can be regarded as a function of n_F and n_R.

Among the geometrical numbers of chief interest we have the number N_1 or α which represents the ratio of the area bounded by the horizontal water line to the area of the rectangle whose sides touch the water line. This is a fullness ratio. There is also a fullness ratio N_2 or β for the cross-section below the water line at a place where the ship has its greatest width. The block or prismatic coefficient

N_3 or δ is the ratio of the volume below the water line to the volume of a circumscribing box. Other numbers are sometimes used. There is the ratio of length to beam L/B which may be called N_{12}, the ratio draft to length which may be called N_{31} and the ratio of beam to draft which may be called N_{23}. The ratio of draft to total height N_{33} and the ratio N_1N_2/N_3 are also of interest. The last ratio is denoted in Germany by the symbol $\varkappa^{-1}$.

4. *The design of a ship.* It is a common practice in engineering to proceed step by step from a good design to a better. If the effect of small variations in the geometrical numbers can be found by tank experiments and are small then Taylor's theorem or the theorem of the mean value can be used to ascertain the effect of a number of simultaneous changes. It sometimes happens, however, that the effect of a small change in a geometrical number is surprisingly large. When the Normandie was being planned[10] and tank tests had been run the owners of the ship wanted more cargo space and a small increase in breadth from 116 feet to 117 feet 8 inches was proposed. Fresh tank tests showed that the resistance of the ship would be increased by at least 6% if the length were kept constant, consequently the length, breadth and height were all increased in the same ratio so as to preserve geometrical similarity.

A resistance coefficient may thus be very sensitive to some changes in geometrical numbers. Certain other changes may have very little effect and when this is the case there is some hope of obtaining optimum values of certain numbers so far as resistance is concerned. Generally, however, many matters have to be taken into consideration in the choice of standard values of the numbers N and a standard form of ship. Lovett[11] has made a statistical study of the ships listed in Lloyd's register for 1921-1922 and gives a table of block coefficients for values of N_{12} ranging from 6.5 to 8.3 and for values of c ranging from 0.5 to 0.7. Sir Amos Ayre[12] says that many designers have obtained good results by using $1.08-\frac{1}{2}c$ as the value of the block coefficient. Lovett states that according to the practice in 1922 the breadth B of a ship 400 feet in length or thereabouts has a limit exceeding $L/10$ by 18 feet while the depth is about 0.63 B and the draft about 81 hundredths of this. He also indicates what changes in form are generally advantageous. Remarks on the dimensions of the most economical ship have been made by J. Foster King.[13]

5. *Equations for the wetted surface of a ship.* It is convenient to take the axis of X in the water plane and to introduce a set of dimensionless co-ordinates x,y,z connected with the original co-ordinates X,Y,Z, representing lengths, by the equations

$$x=2X/L, \qquad y=2Y/B, \qquad z=2Z/D.$$

The familiar form of ship has been evolved from one found to be suitable for a sailing ship. Ample keel surface was provided to prevent side-slipping under the pressure of wind against the sails. This form is still preserved in the yacht the design of which for racing purposes has been greatly developed in this country. A mathematical equation which gives a good idea of the cross section in the yz-plane is that mentioned by Weinblum[14]

$$|y| = (1-z)/(1+mz^3)$$

where m ranges from 7 to 10, the higher value of m giving the sharper point at the keel and the smaller value of N_2.

As steamships began to replace sailing ships this peg-top section, though used by Sir William Symonds,[15] did not fully accomplish the designer's purpose of making the ship easy in a sea way and was generally replaced by an almost rectangular midship section. A departure from the type just mentioned was made by Sir Joseph Isherwood, who invented the arcform ship[16] in which for a given area of cross-section the breadth is about 10% greater at the water line and the bilge is not so pronounced. Comparative tests of the arcform design in British, Dutch German and American tanks agreed in showing an increased efficiency over the box form of about 15%. The simple equation $y=1-z^m$ with, say, $m=4$ is not as good for the usual range of values of n_F as a sack like section represented by an equation of type $|y|=(1-z^8)(1-Cz)$ where Weinblum[14] chooses $C=2/9$ so as to make $N_2=0.8$. The part of the section near the water line is now almost straight and inclined at an angle to the vertical. A section of hollow form half way down and a pronounced bilge may be represented by an equation of type

$$|y| = (1-z^4)(1+3z^4)(1-Cz)$$

where Weinblum[14] chooses $C=1/3$ so as to get a degree of fullness $N_2=8/9$.

For the water line Weinblum mentions the equation

$$|y| = (1-x^4)(1-Cx^m)$$

with $m=2$ or 4. The number N_1 is then $0.8-4C/(m+1)((m+5)$. The slope at the end is $dY/dX=-4B(1-C)/L$.

A simple form for the equation of the surface is $y=f(x)h(z)$ and for this form the number κ is unity. A more general form is that in which y is represented by a sum of products of type $f(x)h(z)$. The section by a plane $y=$constant is also of great importance. In the ships of Sir Edward Harland the length was comparatively great in com-

parison with the beam and depth, also the forefoot was cut away. All this was with the aim of avoiding pitching. The Maier form of ship,[17] which was designed originally for low frictional drag, was also found to be fairly steady in a sea way. In this case the front sections below the water line were almost triangular in shape with their centroids on a straight line inclined to the horizontal. This form was not unlike that of the forebody of a modern racing yacht or of an icebreaker such as the Ermack of Admiral Makaroff, the front of which is required to bear down on a sheet of ice and break it.

A catenary as the cross section in the yz-plane was tried in some experiments at Flushing[18] in 1894. This form is not unlike the sack form already mentioned. The corrugated ship of A. H. Haver[19] had a single groove along each side of the vessel and thus resembles the type with hollow line and pronounced bilge. Some tests indicated very low resistance but their accuracy was questioned. An entirely different way of reducing resistance was suggested to the British Admiralty in 1872 by the Reverend Charles Reade Ramus[20] of the Rye Sussex. His idea of using stepped inclined planes at the bottom of a ship was tested experimentally by W. Froude in some of his early tank experiments. The results were not favorable because the tests were made at low speed but the basic idea ultimately became fruitful when Sir John Thornycroft began to design hydroplanes and when seaplanes and flying boats were developed. The instability of the hydroplane has led to changes in the design of speedboats and Mr. Scott-Paine[21] who has had much experience in design and racing recommends a hard chine section varying in a suitable way over a considerable length. This form has been used for speed boats with which he has made records and for a type of torpedo boat.

My nephew, Mr. Frederick Dodd of the Hughes Hotel, Fresno, who some years ago had much experience in winning races and making new records with speed boats, tells me that the boat used then had a bottom that was absolutely flat except for a non-trip chine to give steadiness in turning. On account of the dynamic lift the front part of the boat was above water at the highest speed, the water line running backwards only about two feet to the stern. The equation of the cross-section of such a boat was thus of type

$$|y| = 1, \quad 0 < z < a$$

$$|y| = 1 - C(z-a), \quad a < z < b$$

$$z = b, \quad |y| = 1 + Ca - Cb \text{ to } |y| = m$$

$$|y| = m - n(z-b), \quad b < z < 1$$

$$z = 1, \quad |y| = m - n(1-b) \text{ to } |y| = 0.$$

A common type of hydroplane had a section represented by the equations

$$z=1, \quad 0<|y|<b$$

$$z=1-m(|y|-b), \quad b<|y|<1,$$

the upper part of the section was generally curved. In the modern seaplane float the flat bottom is practically eliminated, the hard chine beginning at $|y|=0$ and making a fairly large angle with the vertical. The upper part is more or less curved and the sides hollowed out, but the bilge is not generally as marked now as it was formerly.

6. *The Michell theory of wave-resistance.* In 1898 Michell[22] gave a hydrodynamical theory of the wave resistance of a slim ship in which the resistance R was expressed by an integral over the range $1<s<\infty$ of a positive function calculated from certain trigonometrical integrals involving the function $y=f(x,z)$ which gives the form of the surface of the wetted portion of the ship. Some progress in the evaluation of the integral has been made with the aid of Bessel functions of the second kind and their integrals, the results of T. H. Havelock[23] being of chief interest. Havelock also was the first to show that the method can give results at least qualitatively correct in the case of ship forms that are not exactly slim. By choosing the two-dimensional case of a ship of infinite draught and an equation of form

$$|y|=(1-x^2)(a-bx^2)$$

he was able to obtain results agreeing at least qualitatively with the experiments of D. W. Taylor on the hull with bulbous bow.

Havelock also found that by using the Michell integral for the case of the spheroid he obtained results which agreed well with those he had found by another method. These results were also of interest in connection with the theory of the bulbous bow for according to an idea put forward by Froude wave resistance depends upon the amount of interference between the wave trains produced by the bow and stern and when it was found by experiment that the front part of the ship is chiefly instrumental in producing waves engineers toyed with the idea of producing some kind of interference between waves produced at the surface by the stem and below the surface by some kind of a bulge which has been regarded by some as a direct descendent of the ram which was formerly used on the old type of battleship and was retained for some reason when it was no longer needed for its original purpose.

Many experiments with bulbous bows were made about the time of the designing of the Europa, Bremen and Normandie. When, in

fact, the Normandie was being planned,[10] Mr. Yourkevitch, who had made tests at Grenelle, approached the shipbuilders and offered to collaborate in the design of the hull. His offer was accepted and after numerous tests in the Hamburg tank his design was adopted.

Tests at Teddington seem to show that a bulbous bow may be detrimental when the ship is down at the stem and may lead to an increase of drag amounting in some cases to 4% or 5%. In the design of the Queen Mary[24] the shipbuilding firm of John Brown decided not to depart very far from the standard design of ship yet 8000 tests were made with 22 models in the Clydebank tank. In the ultimate design the normal straight stem was adopted but modern tendencies were recognized by the use of a marked forward rake and rounded "fashion plates" at the stem head combined with a pronounced flare of the bow plating above the water line. To overcome the necessity of a very large number of tank tests attempts have been made to make the Michell theory really useful in design. By following up a suggestion made by Michell, W. C. S. Wigley[25] has made calculations for some surfaces of type $y=f(x)h(z)$ in which in two cases $f(x) = \cos \frac{1}{2}mx$ with $m=\pi/8$ and in one case $f(x)=1-\cos mx$. The forms chosen for $h(z)$ were $1-z^2$, 1.333 and $0.32(1+\cos mz)$. The comparison with experiment was fairly satisfactory, the calculated drag was on the average smaller than the experimental, particularly at high speeds; the humps in the drag curve were greater, the hollows still more marked as they corresponded to flat regions in the experimental curve and they occurred earlier.

Impressed by the successes of Havelock and Wigley, G. Weinblum[14] made the Michell theory the basis of a systematic method of approximation in which use was made of numerical methods of evaluating resistance integral after fairly general but simple expressions have been adopted for the equation of the wetted surface of the ship. To facilitate the work the integrals

$$\int_{-1}^{1} \sin(nx)x^m\,dx, \qquad \int_{-1}^{1} \cos(nx)x^m\,dx$$

were tabulated for $m=0(1)9$ and for $n=1(.2)2(.5)12$. In the case of the first integral the range for n was extended up to $n=30$ by steps of 1 and values were also given for $n=2.2$, 2.4, 12.5. Another table gives

$$\int_{0}^{1} e^{-nt}t^m\,dt$$

for $n=0(1)10$ and for values of n ranging from 0 to 6.

Weinblum thus examines the influence of various frame and water lines. He discusses in particular the effect of frame fullness, fore and aft symmetry and the bulbous stem. The effect of fullness of form is also discussed by Kent.[26]

One peculiarity of the Michell theory which is proved rigorously by Hogner[27] is that an asymmetric ship should have the same wave resistance whether it travels forwards or backwards. This is not true for the residual resistance formed by subtracting the skin friction from the total drag and so the disagreement with experiment must be due to the fact that eddy making is ignored in the usual calculations of wave resistance. The effect of thickening the stern has been considered by Weinblum[28] and others. Weinblum has also considered the variation problem giving the surface of least wave resistance.

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The Teacher's Department

Edited by
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Construction of Objective Tests in Mathematics

By RUTH M. BALLARD
Wright Junior College, Chicago

Construction of examinations is one of the ever present problems of a teacher of any mathematics course. The teacher can wait until the day of the examination and write on the board a hastily concocted set of questions which seem to him at the time to cover the subject matter. He may construct the set of questions more carefully either just before the test or, as many books on education advise, early in the teaching period covered by the examination. On the other hand, he may prepare a set of objective questions in mimeographed form.

The objective form of test can be used at any level. Experience has shown that it is suitable and valuable for college courses in mathematics. Tests for recognition of facts, definitions, and formulæ are needed and can be satisfactorily constructed in objective form. An instructor can formulate questions to use as tests for the understanding of proofs, of mathematical properties, and of methods of solving problems as well as questions to use as tests for skills in techniques and in problem solving.

Tests for recogniton or recall can be constructed in any of the usual forms—multiple choice, true false, completion or matching. Multiple choice seems to be particularly usable as in the following examples:

1. Given that x and y are variables, then y is said to be a function of x if (a) x and y are so related that to a given value of x there corresponds one or more values of y, (b) y is always greater than x, (c) y increases as x increases.

2. If $y = \sin^{-1}x$ the values of y are (a) ratios, (b) the reciprocals of the value of $\sin x$, (c) expressed in angular units.

3. The equations of the curve traced by a point on the circumference of a circle which rolls along on a straight line without sliding are

(a) $$x=r(\theta-\sin\theta),\ y=r(1-\cos\theta);$$

(b) $$x=(r+R)\cos\theta-r\cos\frac{r+R}{r}\theta,$$

$$y=(r+R)\sin\theta-r\sin\frac{r+R}{r}\theta;$$

(c) $$x=r\cos\theta+r\sin\theta,\ y=r\sin\theta-r\,\theta\cos\theta.$$

Matching of names of quadric surfaces and their equations, or of polar coordinate equations and their corresponding curves is a good way of testing for the desired information. Lists in which exact matching is possible may be too easy, but if ten names are given and fifteen equations offered, the test is made desirably more difficult. True false can be tried but in many cases in mathematics there seems to be a choice among more than two alternatives. The completion form of question has its advantages, but practically all completion questions can be recast in the multiple choice form if desired.

For testing understanding the completion form is the eminently suitable type. Many objective examinations are entirely recognition tests, but this limitation is not inherent in the form. Questions testing understanding are harder to devise. Whether a given question tests recognition or understanding may vary from student to student. It may depend on just the manner of presenting the material. For instance "The number which has no reciprocal is. . . ." would be a recognition question for a student who has been taught the property of zero in exactly that form. For a student not so taught the answer would require understanding of the definition of reciprocal and of the properties of zero. By giving problems or parts of problems using very simple figures a student's understanding of how to handle a situation can be tested. For instance,

Given the line $$\frac{x-3}{2}=\frac{y+4}{1}=\frac{z-2}{-3}:$$

(a) The line passes through the point. . . . . ., because. . . . .
. . . .

(b) The line is. . . . . .to the plane $2x+y-3z=8$ because. . . . .
. . . .

Steps in processes may be asked for in words. Problems, for instance, involving absolute and relative error are completely thrown off by a misplaced decimal point or other simple mechanical error. Students may understand the college problem and lose all credit for that understanding because of errors in arithmetic. The following question was designed to get around that difficulty and still test whether or not the student understands the process.

The width of a rectangle is $5\pm.2$. Its length is $6\pm.3$. In words

(a) The relative error in width is..............................

(b) The relative error in the area is..........................of the relative errors in the length and width.

(c) The absolute error in the area is the product of...............
.........................and....................

The line of demarcation between skill and understanding is one which is hard to draw. Techniques such as interpolating, using a table of logarithms, differentiating, integrating, and processes which can be performed with no idea of the why and wherefore can be unhesitatingly tagged as skills. However most problems require both understanding and skill. If a problem is analyzed into steps, it can be arranged in completion form. There may be steps in the problem which require recognition and understanding but the problem as a whole can be considered as a test of skill. For instance

Given the polar equation of a conic $\rho=\dfrac{12}{3+\sin\theta}$

(a) The numerical value of the eccentricity is $e=$...., and therefore the conic is.........................

(b) The intercepts are...., ..., ..., ...

(c) The center is at the point (....,....).

(d) The foci are at the points (....,....) and (....,....).

(e) The equation of the directrix is................

(f) Sketch the curve showing intercepts, center, foci and directrix.

Objective tests can be satisfactory both from the point of view of the students and the instructors. An objective test tends to measure a student's achievement at the college level more than his ability in arithmetic. Problems analyzed into steps are intended to give the student the maximum opportunity to go as far as he can with a problem. The scoring should provide an exact and fair system for making the usually vexatious decision of how much credit to award for partly

finished work. In general the scoring and the definiteness of the questions remove most of the opportunities for quibbling about marks. Further, the wide range that it is possible to cover in an objective examination tends to do away with the student's complaint that he knew perfectly all the things he was not asked and had not studied the few things he was asked about.

Objective tests are valuable not only for measuring the accomplishments of the students but of the instructors or the department as a whole. If used by an entire department they tend to make instruction uniform—not in method but in standards for achievement. Objective tests can be used as a norm in colleges where changing world or local conditions cause pronounced variations from year to year. They are of help to the individual instructor in discovering what items he has undertaught, and on what items he is doing a good job. If he tries out new methods of teaching, he can judge their effectiveness. By a study of the tests he can learn how to write better tests and to improve on the ones he is using.

The instructor may feel that an objective test takes too long to construct. This is an objection that is more apparent than real. A non-objective test that is as thorough, covers as much ground, is as well phrased and as well balanced as a good objective test would take many hours to construct. Slip shod test construction can get by more easily on a blackboard written test than it can on a mimeographed objective examination. The labor of preparing an objective test gives or should give an instructor insight into the objectives of his teaching. A wise preliminary to writing the questions is listing his objectives specifically and classifying the learnings he desires as recognitions, understandings, and skills. Such analysis should be an aid to his teaching. Further, because of the time usually required for the mechanical process of mimeographing the test must be prepared well in advance, as is advocated by most authorities. A test once written can be used as a basis of future tests. It can be used later as it is or in revised form. If alternative forms are required, they can be simply provided, as is explained later in this paper. Objec ive tests can be scored quickly. If they are truly objective they can be scored by an assistant. Though objective tests may require a heavy original investment of time, the investment is made where it does the most good and prepays later installments.

Perhaps an objective examination doe; give a student more opportunity to guess, but it gives him less chance to be vague and hard to follow than does a non-objective examination. It may give him less practice in organizing material, but it gives him more drill in

answering specific questions. Copying can be guarded against in either form of examination. If precautions are taken cheating can be detected on objective examinations, and the evidence be conclusive.

The alternative form is the weapon effective against the copier. Tests can be stacked ready to be handed out so that no two students in adjoining seats have the same form. In extreme cases three forms can be used. If there is any suspicion that the students are exchanging forms, the alternation of forms can be checked with the seating plan of the room, and offenders easily located. The ultra careful or suspicious instructor can hand out copies of the test each of which has a student's name written on it. These alternate forms are not hard to construct. If the answers to multiple choice questions are given as letters written in the margin, a mere rearrangement of the order of choices is all that is needed. Numerical problems will have completely different answers if figures are slightly juggled. Such changes should be made with care and the answers worked out to make sure that the alternations have not made one form arithmetically or algebraically more difficult than the other. For instance a slight change in the coefficients of a quadratic equation might change roots from integers to irrationals. Two problems in Horner's method would not be equivalent if one had some zero coefficients and the other did not. If the same question in completion form were wanted on both forms of the examination, it could appear as question 5 on one sheet and question 8 on the other. If such a rearrangement were not desired the question might still be placed at a different spot on the page by allowing different margins at the top and bottom of the pages or by other tricks of spacing. Differently spaced typing of the same question could put a blank at the end of a line in form A and at the beginning of the following line in form B. Any such tactics would be disconcerting to the quick glancer at his neighbor's paper.

Before an alternate form is constructed, however, a first form must be prepared. It is wise to have some general scheme for arranging the questions. At Wright Junior College objective tests in mathematics have been given consistently ever since the opening of that institution in 1934. The tests are designed to have a perfect score of sixty. They are divided into three parts, each with a perfect score of twenty. The recognition questions are usually exactly ten in number and come first. The second part is made up of questions on understanding, but sometimes they are not exactly ten in number since there are subdivisions within the questions. The third part intended to test skill always contains some problems analyzed into steps. When a test sheet is so divided, the test maker must be very careful that he

places questions in the various divisions on the basis of their nature and not on the basis of convenience in phrasing. Just because multiple choice questions have been chosen to test recognition in part one, does not justify including in part one the understanding question that is easily phrased with alternative answers.

In writing the questions care must be taken in the wording. The student's ability to misinterpret is almost beyond belief. If in a completion test a numerical answer is required it is best to ask for it in so many words. For instance,

"The numerical value of the eccentricity of the conic

$$\frac{x^2}{16}+\frac{y^2}{25}=1 \text{ is.....}"$$

If the words "numerical value" are omitted the student could reply c/a or the ratio of the focal distance and the length of the major axis.

The vocabulary should be carefully checked. For example, if the text refers to rectangular coordinates, a reference to Cartesian coordinates is confusing. If the book discusses line symmetry don't ask a question about axial symmetry.

Too many blanks in one sentence tests ingenuity not mathematics. A blank for the first or second word may nonplus the student. In general the nearer the end of the sentence the blank appears, the better. In a multiple choice question the alternative phrases should come at the end of the sentence. If words follow the alternatives it may be difficult to determine whether they apply only to the last phrase or to the whole set of phrases.

An answer may be given away by the singular or plural form of a verb, by an "a" or an "an" preceding the blank, or by some similar slip. If possible such a question should be reworded. If the word arrangement seems unavoidable use some such device as $a(n)$.

Drawings can be used on mimeographed tests. Sketching on stencils is a skill that can be acquired. The added possibilities for good questions include partly completed problems, recognition of curves or properties of curves, the skills in graphing, understanding of symmetry, and locus problems.

Questions should be phrased in a positive manner whenever possible. There should be no tricks or catches in them. Those which test one thing at a time are desirable but not always attainable.

After a first draft is completed, the test should be carefully checked to see that there are no serious omissions nor overemphasis, that there is a balance or at least a rough balance between importance of topics and points dependent on those topics. The problem of marking the

test must be considered and a plan worked out for scoring each question and each part of a question. It is an aid to scoring to label each part of a question (a), (b), and so on.

When the test is considered to be ready it is a good plan to submit it to some other member of the department. Ask him to watch for places where the wording is not absolutely clear or where he could give more than one answer. If the test is being used by a whole department or a large group within the department it may be wise to submit it to several if not all of the instructors who are going to use it. In fact it is highly recommended that the test itself represent the collaboration of two or more of the faculty. Good objective tests are difficult to construct, but are worth the time and labor spent upon them.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SOLUTIONS

No. 369. Proposed by *Dewey C. Duncan*, Los Angeles City College.

Let a, b, c, d be integers with no common factor such that

$$a^2+b^2+c^2=d^2.$$

Prove:

(i) d is odd; of a, b, c, two are even and one odd.
(ii) 12 is the greatest number which always divides $abcd$.
(iii) if 3 divides d, 3 does not divide abc.
(iv) if 3 does not divide d, 9 divides abc.
(v) if $d=4k-1$, 8 does not divide abc.
(vi) if $d=4k+1$, 16 divides abc.

Solution by the *Editors*.

(i) follows immediately from the fact that every odd square is of the form $8k+1$, while every even square is of the form $4k$. Similarly (iii) and (iv) follow because every square is of the form $3k$ or $3k+1$ according as it is or is not a multiple of 3. Thus 4 and 3 are always divisors of $abcd$; the special case $2^2+2^2+1^2=3^2$ shows that no greater number than 12 is a necessary divisor of $abcd$; thus (ii) is proved.

For definiteness let a, b be even and c odd. Then in

$$a^2+b^2=(d-c)(d+c) \tag{1}$$

the factors are both even—say $d-c=2m$, $d+c=2n$, whence $d=m+n$. Since d is odd, one of m, n is even and the other odd. Put

$$(2) \qquad \begin{aligned} d-c&=2u \\ d+c&=4v \end{aligned} \quad \text{or} \quad \begin{aligned} d-c&=4v \\ d+c&=2u \end{aligned}$$

where u is odd. In the Theory of Numbers it is shown that: if p, a prime of the form $4k-1$, is a factor of a sum of two squares, a^2+b^2, then p is a factor of a and of b. Thus p^2 is a factor of both members of (1), and we can show that u in (2) is of the form $4k+1$: for if it were of the form $4k-1$, it would have at least one prime factor p of the same form to an odd power; whence p would divide also $d\pm c=4v$, and be therefore a factor of both c and d as well as of a and b, contrary to the hypothesis. With $v=2h+1$ or $2h$, (2) gives

$$(3) \qquad d=u+2v=4(k+h+1)-1 \text{ or } 4(k+h)+1$$

according as v is odd or even. Next, note that if 4 is a divisor of a or b but not both, then a^2+b^2 is not divisible by 8, contrary to the proof of (i). Thus we have two cases: first, 4 divides neither a nor b, whence a^2+b^2 is divisible by 8 but not by 16, so that v is odd; second, 4 divides both a and b, whence a^2+b^2 is divisible by 16, so that v is even. Comparison of this result with (3) completes the proof of (v) and (vi).

No. 388. Proposed by *Harold S. Grant*, Rutgers University.

Show how to obtain the necessary and sufficient conditions that a polynomial equation of degree n possess a root of multiplicity $m(2<m\leqq n)$ in terms of rational, integral functions of the coefficients.

Solution by the *Proposer*.

If
$$f(x)\equiv\sum_{i=0}^{n} a_i x^i=0$$

possesses an m-fold root; then $f(x)=0, f^{(1)}(x)=0,\cdots, f^{(m-1)}(x)=0$ have this root in common, and conversely, where $f^{(i)}(x)$ denotes the ith derivative of $f(x)\equiv f^{(0)}(x)$. Furthermore, if $f^{(i)}(x)=0$ and $f^{(j)}(x)=0$, $i<j$, have a common root, then $R(x)=0$ also possesses this root, where $R(x)$ is the remainder when $f^{(i)}(x)$ is divided by $f^{(j)}(x)$. Conversely, if $R(x)=0$ and $f^{(j)}(x)=0$ have a common root, this root must satisfy $f^{(i)}(x)=0$. To find the necessary and sufficient condition, then, that the literal equations $f^{(i)}(x)=0$ and $f^{(j)}(x)=0$ have a common root we go through the same procedure as in finding the greatest common divisor of $f^{(i)}(x)$ and $f^{(j)}(x)$ until the last divisor is linear and the re-

mainder constant. The linear divisor will give us the root as a rational function of the coefficients, and the remainder, equated to 0, the required necessary and sufficient condition. We do this for the $m-1$ pairs of equations

$$f(x)=0,\ f^{(m-1)}(x)=0;$$

$$f^{(1)}(x)=0,\ f^{(m-1)}(x)=0;\cdots;\ f^{(m-2)}(x)=0,\ f^{(m-1)}(x)=0;$$

obtaining thus $m-1$ conditions from the vanishing of the remainders and $m-2$ conditions from equating the roots. This constitutes a maximum of $2m-3$ equations of condition.

For example, taking $n=4$ and $m=3$, we obtain $Ax+B$ and $Cx+D$ as the linear divisors from $f(x)$, $f^{(2)}(x)$ and from $f^{(1)}(x)$, $f^{(2)}(x)$, respectively, in which

$$A=8a_1a_4^2-4a_2a_3a_4+a_3^3, \qquad C=8a_2a_4-3a_3^2,$$

$$B=72a_0a_4^2-10a_2^2a_4+3a_3^2a_2 \qquad D=6a_1a_4-a_2a_3.$$

The remainder of $f(x)$ and $f^{(2)}(x)$ gives

(1) $$27a_2A^2-9a_3AB+2a_4B^2=0,$$

and of $f^{(1)}(x)$ and $f^{(2)}(x)$ gives

(2) $$a_2C^2-3a_3CD+6a_4D^2=0;$$

and equating the two roots gives

(3) $$9AD=BC.$$

But it will be noticed that if (1) is multiplied through by C^2 and (2) by $27A^2$, the results are identical by virtue of (3). Thus the three conditions are not independent if $A\neq 0$, $C\neq 0$. On the other hand $A=0$ in (1) and $C=0$ in (2) imply $B=0$ and $D=0$ respectively, since $a_4\neq 0$; so that in all cases, either (1) or (2) or (3) is superfluous.

In general let $Cx+D$ be the linear divisor obtained in finding the greatest common divisor of $f^{(m-2)}(x)$ and $f^{(m-1)}(x)$ and suppose that $C\neq 0$; then the $m-1$ conditions obtained by eliminating x between $Cx+D$ and the $m-1$ equations $f^{(m-2)}(x)$, $f^{(m-3)}(x),\cdots, f^{(1)}(x)$, $f(x)$ are both necessary and sufficient. To handle the case $C=0$ and obtain a definite set of equations as the necessary and sufficient conditions presents considerable difficulty unless we treat it as in the first paragraph above. However, it seems in general that $m-1$ conditions are needed; additional conditions are not independent.

No. 413. Proposed by *Paul D. Thomas*, Southeastern State College, Oklahoma.

A circle is described on the line joining the foci of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

as a diameter. Show that the locus of the pole of a variable chord of the hyperbola which is tangent to the circle is also the envelope of the chords of contact of pairs of orthogonal tangents drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Solution by *Gerald B. Huff*, Southern Methodist University.

Using the usual $c=\sqrt{a^2+b^2}$ as the distance from the center of the hyperbola to the foci, we know that

$$\frac{x\cos w}{c}+\frac{y\sin w}{c}=1 \tag{1}$$

is the equation of a variable tangent to the circle. Since the polar line of $(\bar{x},\bar{y})$ with respect to the hyperbola is

$$\frac{x\bar{x}}{a^2}-\frac{y\bar{y}}{b^2}=1, \tag{2}$$

if $(\bar{x},\bar{y})$ is the pole of a tangent to the circle, then

$$\frac{\bar{x}}{a^2}=\frac{\cos w}{c}, \quad \frac{\bar{y}}{b^2}=-\frac{\sin w}{c}. \tag{3}$$

These equations are the parametric equations of the locus of the poles of the tangents to the circle.

To obtain a set of parametric equations of the envelope of the chords of contact of pairs of orthogonal tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we first remark that $x^2+y^2=c^2$ is the director circle of the ellipse; i. e. it is the locus of points from which pairs of orthogonal tangents may be drawn. The polar line of a point on this circle is a chord of contact

of two such tangents. Since ($c \cos \Theta$, $c \sin \Theta$) is any point on this circle, all such chords are given by

$$(4) \qquad -\frac{xc \cos \Theta}{a^2}+\frac{yc \sin \Theta}{b^2}=1.$$

The parametric equations of the envelope are obtained by solving (4) and the equation obtained by differentiating with respect to Θ:

$$(4') \qquad \frac{xc \sin \Theta}{a^2}+\frac{yc \cos \Theta}{b^2}=0.$$

Solving for $\dfrac{x}{a^2}$ and $\dfrac{y}{b^2}$ yields:

$$(5) \qquad \frac{x}{a^2}=\frac{c \cos \Theta}{c^2}, \quad \frac{y}{b^2}=\frac{c \sin \Theta}{c^2}.$$

The loci represented by (3) and (5) are evidently the same.

Also solved by *C. W. Trigg* and the *Proposer*.

No. 415. Proposed by *Richard L. Seidenberg*, student, Colgate University.

A design is to be made showing the "Red Cross" inscribed to a given circle. What must be the dimensions of the cross so that the area is a maximum. Give a geometric construction.

Solution by *Leon Shenfil*, student, Los Angeles City College.

Let the dimensions of the two rectangles forming the cross be x, y with $x>y$ and $x^2+y^2=4r^2$. The area of the cross is

$$A=2xy-y^2,$$

or, if $\tan \Theta=y/x$ (2Θ the central angle subtended by an edge y of the cross),

$$A=4r^2(\sin 2\Theta-\sin^2\Theta).$$

Setting $dA/d\Theta=0$, there results for a maximum

$$\tan 2\Theta=2.$$

The construction of the angle 2Θ produces the cross. Dimensions of each rectangle are approximately $x=(1.701)r$, $y=(1.051)r$.

Also solved by *D. L. MacKay* and the *Proposer* who notes that the dimensions are associated with the Golden Section, for it is true that

$$y/x = x/(x+y).$$

No. 417. Proposed by *Paul D. Thomas*, Southeastern State College, Oklahoma.

Show that the family of curves which satisfies $(ds/dx)^2 = R$, where ds is the element of arc and R the radius of curvature is a two parameter family of catenaries.

Solution by the *Proposer*.

$$\frac{ds}{dx} = (1+y'^2)^{1/2}, \quad R = (1+y'^2)^{3/2}/y''.$$

Hence $1+y'^2 = (1+y'^2)^{3/2}/y''$ which becomes

$$y''/(1+y'^2)^{1/2} = 1. \tag{1}$$

Place $y' = p$, $y'' = pdp/dy$ then (1) is $pdp/(1+p^2)^{1/2} = dy$. Integrating, $(1+y'^2)^{1/2} = y+a$, and rationalizing $dy/[(y+a)^2-1]^{1/2} = \pm dx$. Integrating, $\cosh^{-1}(y+a) = \pm x+b$ or $y = \cosh(x+b)-a$, a two parameter family of catenaries.

No. 418. Proposed by *G. W. Wishard*, Norwood, Ohio.

The sum of the cubes of the first n natural numbers is the sum of the first P odd natural numbers: the sum of the cubes of the first n odd natural numbers is the sum of the first Q natural numbers. Prove these statements for every n, and determine P and Q.

Solution by *Morris Chernofsky*, student, Yeshiva College, New York City.

It is shown by well-known, elementary methods that:

(a) the sum of the cubes of the first n natural numbers is

$$(n^2+n)^2/4.$$

(b) the sum of the first P odd natural numbers is P^2,

(c) the sum of the cubes of the first n odd natural numbers is

$$2n^4-n^2,$$

(d) the sum of the first Q natural numbers is $(Q^2+Q)/2$.

Thus the first part of the statement is true if $P=(n^2+n)/2$, and the second part is true if $Q=2n^2-1$, for every n.

Also solved by *Albert Farnell* and the *Proposer*.

PROPOSALS

No. 436. Proposed by *W. V. Parker*, Louisiana State University.

Find the equation of the curve tangent to the circle $x^2+y^2=a^2$ at $(0,a)$ such that its tangent at (h,k) is parallel to the tangent to the circle from $(0,k)$.

No. 437. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find a ten-digit number which will equal the square of the sum of the two numbers formed by its first five digits and its last five digits.

No. 438. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given circle O and point P on the diameter BA prolonged. Construct the secant PDC so that the quadrilateral $ABCD$ is a maximum.

No. 439. Proposed by *Walter B. Clarke*, San Jose, California.

Given angle C. Construct the triangle ABC in which $b-a=hc/2$.

No. 440. Proposed by *H. T. R. Aude*, Colgate University.

A point interior to a square is at the unequal distances a, b and c units from three of the vertices of the square.

(1) Determine the conditions on the three numbers a, b and c which allow the existence of only one square, only two squares, of three squares.

(2) Let the numbers a, b, c be restricted to integers. Of all the admissible sets a, b, c in the three cases cited in (1), find in each case the set a, b, c where the sum $a+b+c$ is the least.

(This problem was suggested by Problem No. 407 proposed by the Mathematics Club of Tulane University and is respectfully dedicated to the members of the Club.)

No. 441. Proposed by *Walter B. Clarke*, San Jose, California.

The chords AB and CD of a circle meet in E. Locate D such that either C, D, or E is the midpoint of the line determined by the other two.

Bibliography and Reviews

Edited by
H. A. SIMMONS and JOHN W. CELL

Junior Mathematics, Books 1, 2 and 3. By Harl R. Douglass and Lucien B. Kinney. Henry Holt and Company, 1940. Book 1, vii+440 pages; Book 2, vii+439 pages, and Book 3, vii+504 pages.

Curricular changes in any subject are reflected in its textbooks. Not so many years ago the traditional course of study in mathematics called for *Applications of Percentage* in grade seven, *Mensuration* in grade eight and *Algebra* in grade nine. From this rigid, uninteresting and prosaic practice we have advanced to a flexible, interesting and stimulating program if one is to judge it by its reflection in the *Junior Mathematics* series of Douglas and Kinney, which is intended for use in grades seven, eight and nine of schools where algebra is begun in grade ten. These authors have written their books with the pupil ever in the foreground; they are continually aware of his interests, abilities, needs and growth.

A glance at some of the chapter headings indicates the wide range of subject matter which will certainly stimulate and maintain interest: Getting Ready for Life with Mathematics, How We Use Percentage in Home and Business, Mathematics and Earning Money, Mathematics and Travel, Mathematics and the Consumer, How We Use a Bank, Mathematics and Buying and Selling, Personal Credit, etc. The excellent illustrations, cartoons, diagrams and pictures contribute to this interest factor and also to clarity and understanding. For variety as well as interest, drill material has been arranged in the form of "Shock Absorbers", "Try These", "Practice Tests" and "Hurdles."

There is quantitative and qualitative provision for individual differences. Diagnostic tests with norms appear at intervals; for pupils who exhibit the need for further drill and review, there is a special chapter, "Diagnostic and Remedial Program", which provides practice according to their particular weakness. There are supplementary topics and problems which are starred to indicate that they are suitable for accelerated pupils or for pupils who are especially interested in that particular topic. The average pupil who studies the *Junior Mathematics* series should show better than average growth. The continual emphasis on problem solving should aid the pupil in emerging from the course not only a better thinker but better prepared to meet the problems of life. One observes with satisfaction the emphasis on accurate computation, the emphasis on checking and the emphasis on the "Why" rather than the "How".

Certain criticisms must be made. In Book 1, page 15, we find the pupil asked to check multiplication by "reversing the numbers"; "interchanging" would be more accurate. The treatment of approximate number and "rounding off" is weak: we find the pupil asked to "round off" early in the book but no explanation given before page 309. The word "compasses" first appears in an exercise with no explanation. On page 389 we find the undesirable statement that 12 ft. x 10 ft. = 120 sq. ft. In Book 3 page 236, we read: "If Angle A is 37°, we see from the table that side a is 0.75 side b"; it is doubtful whether many trained mathematicians are able to view a table of tangents with such penetration let alone the immature ninth grader.

The preview, a recommended pedagogical procedure, is used advantageously by the authors to open each chapter and to give a picture of the work to follow. The frequent use of the developmental approach to a new topic and the use of discussion questions are most commendable. Outstanding are the chapters on Mathematics in Communication, Mathematics and Transportation, and Geometry; equally excellent are the sections on the introduction of formulas, rigidity of triangles, symmetry, parcel post and express. The chapter tests are very good features of the series.

In the report of the Committee on the Function of Mathematics in General Education is the statement: "The purpose of general education is to provide rich and significant experiences in the major aspects of living, so directed as to promote the fullest possible realization of personal potentialities, and the most effective participation in a democratic society." Douglass and Kinney have achieved this purpose in their *Junior Mathematics*. The boys and girls who study these books should be better citizens of this democracy because of their vital experiences with many of the problems of our contemporary civilization.

University High School MILES C. HARTLEY.
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Mathematics for Today, Books 1 and 2. By Harl R. Douglass and Lucien B. Kinney. Henry Holt and Company, 1940. Book 1, vii+437 pages; Book 2, vii+447 pages.

Everyday Mathematics. By Harl R. Douglass and Lucien B. Kinney. Henry Holt and Company, 1940. vii+503 pages

Mathematics for Today was written for grades seven and eight in schools where algebra is begun in grade nine and *Everyday Mathematics* was written for grade nine and the pupils who should not take formal algebra; the three volumes are essentially the same as *Junior Mathematics*. The differences occur in certain omissions, additions and rearrangements: the omission from *Junior Mathematics*, Book 3, of the chapters on Using the Right Triangle in Measurement, What Algebra Is and Some of Its Uses, and What Geometry is Like; the introduction of two other chapters, Using Formulas in Life and Savings and Investments, in *Mathematics for Today* (Book 2) and Mathematics in Providing a Home in *Everyday Mathematics*. The order of chapters has been changed slightly: for example, Chapter V (Book 1) of the first series, Mathematics and Earning Money, becomes Chapter II (Book 2) of the second series, Chapter III (Book 2) of the first series, Expressing Ideas with Lines and Angles, becomes Chapter VII (Book 1) of the second series. These changes in sequence and others not mentioned are indicative of the flexibility and adaptibility of the material presented.

In their introduction the authors state that they have met the recommendations of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics as presented in the *Place of Mathematics in Secondary Education*. They have attempted to provide for the weaknesses in arithmetic. They have given special attention to social problems. They offer a broad mathematical training and outlook. They provide training in arithmetic, graphic presentation but no adequate training in algebra, geometry or numerical trigonometry. In this respect they fail to meet the recommendations of the Commission. Book 3 of *Junior Mathematics* is adequate in this respect. For this reason it would seem that many schools will prefer to use *Junior Mathematics* even though the second series meets the other recommendations of the Commission.

University High School MILES C. HARTLEY.
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Units of Work and Centers of Interest in the Organization of the Elementary School Curriculum. By Sadie Groggans. Contribution to Education, No. 803, Bureau of Publications, Teachers College, Columbia University, 1941.

This Ph.D. dissertation from Teachers College is an attempt to evaluate and reinterpret educational principles which guide curriculum development in the elementary schools. Re-examination of the bases for differences among educators and investigation of present divergencies in curriculum practices result in a carefully documented report that should help the reader to evaluate his own philosophy of education. The appended bibliography of 339 titles refers to much of the choice literature on the elementary school and curriculum building.

The main thesis of the study is that there are two antithetical points of view in education which are expressed in school curricula as *Units of Work* or *Centers of Interest.* The analysis of this thesis is set forth in chapters with the following titles: Antithetical Organizations of Curriculums, Significance of Interpretations Upon the Curriculum, Rapprochement of Teaching and Learning, Priority of Thought and Experience, Education for Conservation and Improvement of Society. A concluding chapter entitled, Organization of the Curriculum and Reorientation of Education Within the Cultural Pattern, apparently accepts the "Center of Interest" idea of curriculum development as the most promising for today's schools.

In the reviewer's opinion, this investigation contributed considerably to the student's clarification of a point of view in education but does not add significantly to curriculum development and elementary school literature. Most of the educational concepts and conclusions are more clearly set forth in some of the original sources that the investigator used. Essentially the study is a recapitulation of educational thinking confused by a questionable attempt to classify curriculum principles and practices in two diametrically opposed schools, of curriculum building which are labeled "units of work" and "centers of interest". Current usage of these terms so confuses their meaning as to make them useless for classification purposes. Furthermore the issues and conclusions are cloaked in undefined educational terms which confuse interpretations and results.

It happens that the reviewer subscribes to the educational philosophy that Dr. Groggans apparently finds acceptable. The reviewer has no doubt that teachers and administrators would provide better schools for children if they fully understood and applied the curriculum principles evolved. He hopes the author will prepare a book *for teachers* now that Ph.D. hurdles are cleared and professional respectibility has been established.

Northwestern University. WALTER A. ANDERSON.

Intermediate Algebra. By C. H. Mergendahl and T. G. Walters. New York, D. Appleton-Century Company, 1941. viii+436 pages. $1.48.

Throughout the book processes are explained in detail not as rules, but as applications of fundamental principles. The function concept is well developed through the formula, variation, and graphs of both linear and quadratic equations.

The content is that of the conservative textbook of the past with the addition of many new problems applied to current affairs. There is a cumulative review at the end of every chapter. All the usual topics of first year algebra are reviewed and extended in the first six chapters. There is a chapter on statistics. The book contains some historical information.

The number of verbal problems and practice exercises is very large. The authors suggest that the teachers are to select the practice exercises as needed and that pupils should form the equations for many of the verbal problems but solve only part of them. This book will not suit the teacher who attempts to "cover the book" and the emphasis on factoring and fractions will be especially unfortunate.

The typography of the book is very good. There are quite a number of unusually attractive photographs which help the pupil to visualize the applications of algebra.

University High School
State University of Iowa. RUTH LANE.

Calculus (Part I). By H. B. Phillips. Lew A. Cummings Company, Cambridge, Mass., 1940. vi+229 pages.

This is a text apparently designed for a rigorous first course in Calculus for students who have had Algebra and Trigonometry. A previous knowledge of Analytics is not assumed, and a major portion of the essential material from this subject is introduced in the chapter on Algebraic Equations and graphs, which is mostly devoted to a survey of conics and curve sketching. However a certain degree of mathematical maturity would be most desirable if the student is to gain full appreciation of the rigor with which the Calculus is developed in this text.

In his first chapter the author gives enough of the theory of sequences and infinite sets to allow thorough proofs of theorems in Calculus which are disposed of rather loosely in most elementary texts. The epsilon-delta method of defining limits and continuous functions is used freely.

The definition of the derivative is followed closely by the introduction of the differential, and formulas for finding the differentials of algebraic functions, and sines and cosines, are developed and featured as such rather than in the usual derivative forms in which most books give them first. Higher derivatives and indeterminate forms are introduced early.

In accordance with the latest trends, integration is introduced in the third chapter, and differentiation of inverse trigonometric functions, logarithmic functions, and exponential functions is reserved for later chapters. Applications of integration are made in finding areas, volumes of solids of given area of section, and water pressure.

One entire chapter is devoted to properties of determinants and the use of determinants in solving sets of linear equations. Curvature is disposed of very briefly in the chapter on parametric equations. Polar areas are given one section in a chapter on polar coordinates. The text closes with a chapter on vectors and their use in the study of curvilinear motion.

The author has evidently reserved such topics as approximate methods of evaluating definite integrals, integration of more complicated forms, trigonometric substitution, integration by parts, series, and partial derivatives for a later volume.

All of the problems in the text are bunched at the ends of the chapters, a feature which does not appeal to the reviewer. The book would be more teachable if more problems were included for purposes of drill. The printing is good but not distinctive. The figures are not outstanding and frequently leave something to be desired in the matter of labeling. There is a clear index, and no answers are given. The reviewer feels that in most colleges the volume would serve better as a reference book than as a class text.

North Carolina State. R. C. BULLOCK.

Tools, a Mathematical Sketch and Model Book. By Robert C. Yates. Louisiana State University Press, Baton Rouge, 1941. 194 pages. $1.60.

This book is a distinct departure from the usual college geometry text. It is not a study of modern geometry nor a review of high school geometry; not a history of geometry nor a professional course for the training of teachers; yet all these features are embodied in it. The book has as its theme geometric tools. It begins with a presentation of the simple Euclidean tools the unmarked straight edge and compasses, with discussions of their constructional possibilities It progresses thence into a study of other modified and more or less complex geometrical instruments, including the parallel ruler, the angle ruler, quadratic and quartic tools, and a variety of plane linkages. Interspersed throughout are timely reviews of fundamental theorems, not only of Euclidean origin, but also from such sources as Menelaus, Ceva, Mascheroni, and Desargues. It is unmistakably the product of vast reading and research, coupled with careful organization and discrimination on the part of its author. It offers in one binder a variety of materials which previously one would have searched through many volumes and treatises to find. It offers in addition many new problems and new ideas of the author's own.

The book is in the form of a loose-leaf workbook and laboratory manual, making it convenient to transfer sheets to other standard loose-leaf binders, or to insert additional sheets. There are approximately 80 plates, containing about 500 drawings, each faced by explanatory text. Some of the drawings serve as problems, others as guides to the construction of cardboard models. Space is provided in the book itself for the student's work, to the end that the student will in the words of the author, "develop the feeling of being co-author of a volume that may serve him later as a source of supplementary material in his career as a teacher." There are suggestions offered to the instructor as to the conduct of a course based upon these materials, such as a suitable division of time between classroom and laboratory.

No prerequisites beyond freshman college mathematics are listed, but it is obvious that the value of the book as a text "can be realized only by some thought and much labor" on the part of both student and instructor. This seems to imply the desirability of maturity and seriousness of purpose. Though the geometrical tool is the point of emphasis throughout, unquestionably the chief worth of the book lies in the providing of a clearer insight into, and deeper appreciation of, geometrical thought and structure, which, in the mind of the writer, is the most desirable type of professional preparation.

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